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ABSTRACT

This guide represents the final experimental version of an extended pilot project which was conducted in the United States between 1973 and 1976. The manner of presentation and the pedagogical ideas and tools are based on the works of Georges and Frederique Papy. They are recognized as having introduced colored arrow drawings ("papygrams") and models of our numeration system (the Papy "minicomputer") into the teaching of mathematics at the elementary and secondary level in Belgium. The CSMP curriculum follows the "spiral approach." The series emphasizes that a topic may provide children with intuitive leaps which might help them acquire successive pieces of information. An introduction is given to the Language of Strings and Arrows. In the String Game, students are to deal with divisors of numbers. A chapter on logical thinking deals with precise terms in English used to discuss statements concerning various sets. Composition Games reiterates what the students experienced earlier in CSMP. Geometry and Measurement and Probability and Statistics are other topics within the guide. (Author/SA)

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Comprehensive School Mathematics Program
CEMREL, Inc.

Experimental Version

4-20601

CSMP Mathematics for the Intermediate Grades

Part I

Teacher's Guide

The Languages of Strings and Arrows
Geometry and Measurement
Probability and Statistics

U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE
NATIONAL INSTITUTE OF EDUCATION

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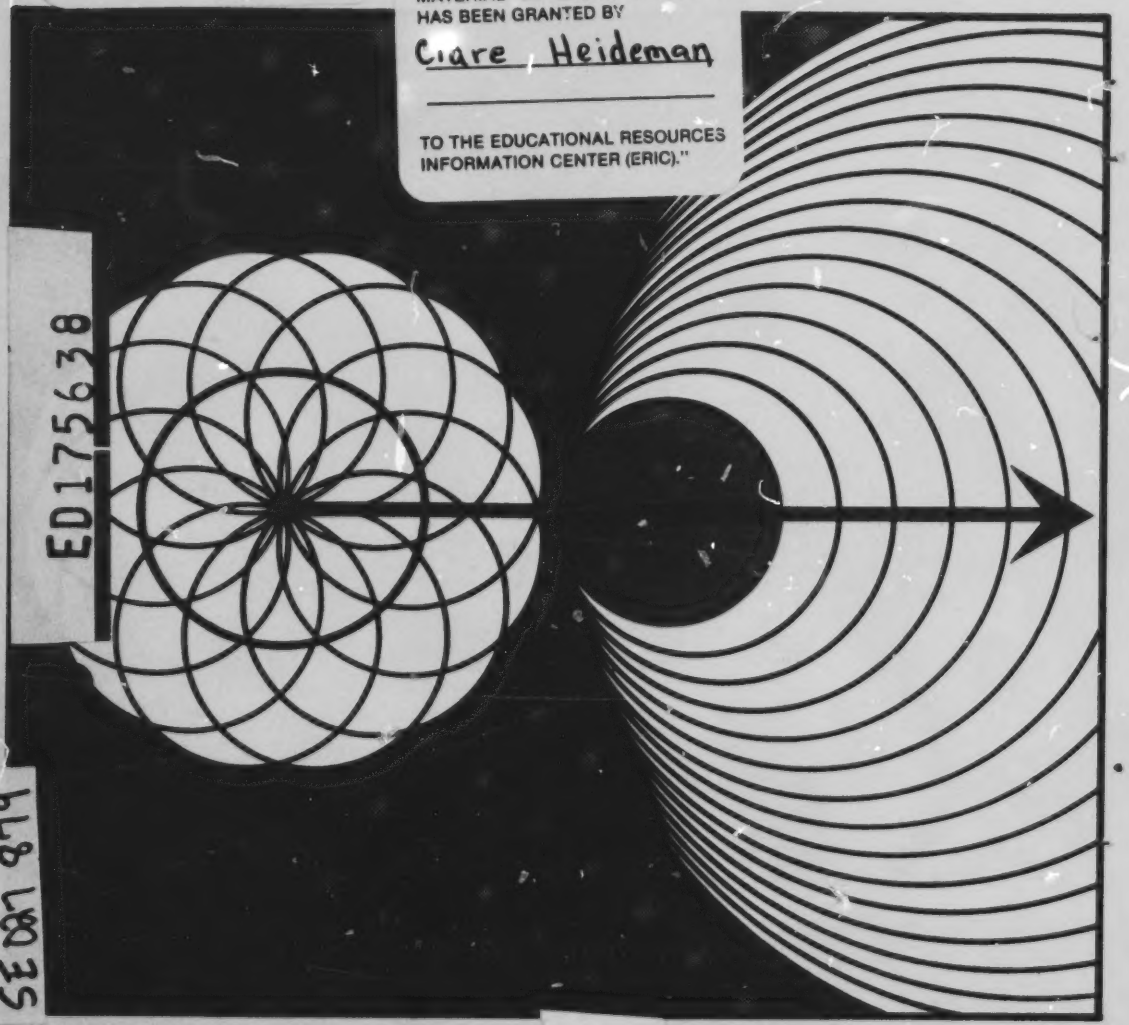

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The Languages of Strings and Arrows

Comprehensive School Mathematics Program

Mathematics for the Intermediate Grades Part I

The Languages of Strings and Arrows

CEMREL, Inc.

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SUGGESTED SCHEDULE FOR TUESDAY'S AND THURSDAY'S LESSONS

WEEK	TUESDAYS	MATERIALS
1	1L L1.1: The String Game with A-Blocks p. 2	Teacher: Colored chalk; The String Game; 2 hand-calculators (optional)* Student: None
2	2L L1.2: The String Game with Numbers: Multiples and Order p. 12	T: Colored chalk; The String Game S: None
3	3L L2.1: Logical Thinking #1 p. 52	T: Colored chalk; The String Game S: Worksheets L01, L02; pencils
4	4L L2.2: Logical Thinking #2 p. 61	T: Colored chalk; The String Game S: Pencils; paper; Worksheets L03, L04
5	4G G2.1: Circles #1 p. 39	T: Compass; straightedge; colored chalk S: Compass; 30 cm ruler; colored pencils
6	5G G2.2: Circles #2 p. 43	T: Compass; straightedge; colored chalk S: Compass; 30 cm ruler; colored pencils
7	6G G2.3: Circles #3 p. 47	T: Compass; straightedge; colored chalk S: Compass; 30 cm ruler; colored pencils
8	FREE DAY	
9	7G G1.4 Area Problems p. 29	T: Grid board; Flip flop aid; colored chalk S: Worksheets G06, G07; colored pencils

*Hand-calculators are used only for the alternate way of presenting exercise 3 in this lesson (see page 7).

SUGGESTED SCHEDULE FOR TUESDAY'S AND THURSDAY'S LESSONS

THURSDAYS	MATERIALS
1G G1.1: Comparisons of Lengths and Areas p. 3	T: Grid board; colored chalk S: 30 cm rulers; Worksheets G01, G02; colored pencils
2G G1.2: Introduction to Millimeters p. 12	T: None S: 30 cm rulers; Worksheet G03
3G G1.3: Square Regions p. 19	T: Four colored squares; grid board; colored chalk S: Colored pencils; Worksheets G04, G05
FREE DAY	
1P P1.1: Same or Different #1 p. 2	T: Marbles; colored chalk S: Marbles; pencils; paper
2P P1.2: Same or Different #2 p. 11	T: Marbles; colored chalk S: Pencils; paper
3P P1.3: Same or Different #3 p. 16	T: Colored chalk S: Pencils; Worksheets P01, P02, P03, P04, P05, P06, P07
4P P1.4: Same or Different #4 p. 22	T: Marbles; colored chalk S: Marbles; colored pencils; pencils; paper
5P P1.5: Same or Different #5 p. 30	T: Marbles; colored chalk S: Marbles; colored pencils; pencils; paper

SUGGESTED SCHEDULE FOR TUESDAY'S AND THURSDAY'S LESSONS

WEEK	TUESDAYS	MATERIALS
10	5L L3.1: Composition Games #1 p. 80	T: Colored chalk S: Colored pencils; Worksheets L09, L10, L11
11	6L L3.2: Composition Games #2 p. 90	T: Colored chalk S: Colored pencils; Worksheets L12, L13, L14
12	8L L3.3: Composition Games #3 p. 99	T: Colored chalk S: Colored pencils; paper
13	10L L3.4: Composition Games #4 p. 105	T: Colored chalk S: Colored pencils; paper; pencils
14	FREE DAY	
15	FREE DAY	
16	6P P2.1: "The Square Trap" #1 p. 36	T: "The Square Trap"; Minicomputer Set S: "The Square Trap"; Worksheet P08; spinner arrows; Minicomputer checkers; pencils
17	7P P2.2: "The Square Trap" #2 p. 38	T: "The Square Trap"; Minicomputer Set S: "The Square Trap"; Worksheet P08; spinner arrows; pencils; paper
18	8P P2.3: "The Square Trap" #3 p. 43	T: "The Square Trap"; Minicomputer Set S: "The Square Trap"; Worksheet P08; spinner arrows; pencils; paper

SUGGESTED SCHEDULE FOR TUESDAY'S AND THURSDAY'S LESSONS

THURSDAYS	MATERIALS
FREE DAY	
7L p. 67 L2.3: Logical Thinking #3	T: Colored chalk; The String Game S: None
9L p. 72 L2.4: Logical Thinking #4	T: Colored chalk; The String Game S: Worksheets L05, L06, L07, L08; pencils
FREE DAY	
8G p. 51 G2.4: Circles #4	T: Compass; straightedge; colored chalk S: Compass; 30 cm rulers; colored pencils
9G p. 59 G2.5: The Triangle Inequality	T: Compass; straightedge; colored chalk S: Compass; 30 cm rulers; colored pencils
11L p. 14 L1.3: The String Game with Numbers: Multiples, Divisors and Order #1	T: Colored chalk; The String Game S: Pencils; paper; colored pencils
12L p. 21 L1.4: The String Game with Numbers: Multiples, Divisors and Order #2	T: Colored chalk; The String Game S: Colored pencils; pencils; paper
13L p. 27 L1.5: The String Game with Numbers: Multiples, Divisors and Order #3	T: Colored chalk; The String Game S: Colored pencils; pencils; paper

SUGGESTED CHRONOLOGICAL ORDER OF "L" LESSONS

Lesson One:	L1.1 - The String Game with A - Blocks	2
Lesson Two:	L1.2 - The String Game with Numbers: Multiples and Order	12
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Lesson Ten:	L3.4 - Composition Games #4	105
Lesson Eleven:	L1.3 - The String Game with Numbers: Multiples, Divisors and Order #1	14
Lesson Twelve:	L1.4 - The String Game with Numbers: Multiples, Divisors and Order #2	21
Lesson Thirteen:	L1.5 - The String Game with Numbers: Multiples, Divisors and Order #3	27

THE LANGUAGES OF STRINGS AND ARROWS

INTRODUCTION

The language of strings represents, at a nonverbal level, the language of sets, that is, the grouping together of objects whether the objects are numbers or not. The language of arrows, on the other hand, is the nonverbal language of relations among objects of the same or different sets. Each of these languages permeates the different content strands. They are sometimes used separately, sometimes together to present an unending variety of numerical and nonnumerical situations to the student.

Classification: The Language of Strings

Classifying things is one of the most important and all-pervasive of human intellectual activities. It is one of the fundamental tools we use to get a handle on things—to keep them straight in our minds, and to avoid being overwhelmed by a sea of chaos. We classify all the time in our everyday life—often unconsciously or, at least, without much deliberate thought: we classify cars by brand (Ford, Chevrolet, etc.); we classify people by citizenship (American, English, German, Australian); we classify the letters of the alphabet into vowels and consonants, and so on. On a more systematic level, much of science is the search for apt and useful systems of classification: the astronomer sorts out the heavenly bodies into stars, planets, moons, comets; the zoologist sorts out animals into reptiles, birds, mammals, insects, protozoa, etc., and the botanist does the same with plants; the linguist sorts out languages into Germanic, Romance, Slavic, etc. Last but not least, the mathematician is always classifying—he sorts out numbers into even and odd; he categorizes geometric figures (by the number of sides) into triangles, quadrangles, pentagons, etc.; he distinguishes between rational and irrational numbers, and so on.

It is true that our penchant for classification is often misused and abused. Educators, for example, sometimes classify children into "bright" and "slow" when it is by no means clear that this kind of sorting out does not oversimplify a very complex situation and does not serve to bring useful insights. We are all aware of the mischief that simpleminded or even vicious stereotyping of people, ideas, or things can work. However, not only the legitimate uses, but even the abuses of identification schemes point to the importance of teaching children to become proficient at handling the ideas behind classifications. Indeed, if the elementary theory of sets has any place at all in the elementary mathematics curriculum, it is precisely because that theory helps us to understand the basic ideas of classification. It is to this purpose that the language of strings is utilized in the CSMP curriculum.

Classifying means just what it says—putting things into classes, or as the mathematician says: sets. In its simplest context a given thing either belongs to a given set (i.e., has a given property) or it doesn't. For example, since Abraham Lincoln was a President of the United States, he belongs to the set of American Presidents, while Spiro Agnew does not belong to that set. The mathematician abbreviates this by saying: Let P be the set of Presidents of the United States; let ℓ be Abraham Lincoln, let a be Spiro Agnew. Then,

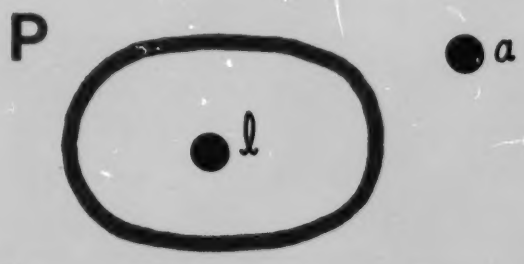
$$\ell \in P \qquad \text{and} \qquad a \notin P$$

where the symbol "$\in$" is read "belongs to" or "is a member of". Similarly, if E is the set of even numbers and O is the set of odd numbers, we have

$$5 \notin E, \quad 5 \in O, \quad 6 \in E, \quad 6 \notin O.$$

The basic idea is incredibly simple: Given a set S and any thing x, either x belongs to S or x doesn't belong to S, either x is "in" S or x is not "in" S; i.e., either $x \in S$ or $x \notin S$. It is the most basic act of sorting—"in" or "out". We can represent this elementary situation in a neat way by using the so-called "string pictures", which illustrate the idea of "in" or "out" in

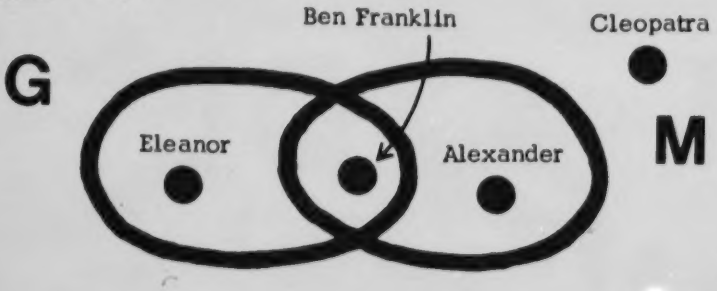
a very dramatic way. We represent the objects (the things) to be sorted by dots, and we represent the sets into which they are to be sorted by drawing "strings". Consider the Lincoln-Agnew situation referred to above. The string labeled "P" represents the set of Presidents. The dots represent various people. Note: Lincoln is "in" P ($l \in P$) and Agnew is "out"; i.e., not in P ($a \notin P$).



Note: A dot in itself can stand for any thing, e.g., Napoleon, the Eiffel Tower, the number 5, the city of Chicago, and so forth. But, once the dot goes inside the P-string it no longer can stand for just anything. It can't represent the Eiffel Tower or Louis Armstrong; it must be for a President of the United States. Of course that still leaves (as of 1976) 38 choices (from George Washington to Gerald Ford). Similarly, if a dot is outside the string it could be for the Eiffel Tower, or the Soviet Union, or the planet Mars, or even the number 5, but it cannot be for Thomas Jefferson.

The situation becomes interesting when we are sorting with respect to more than one property, i.e., when we are putting things into several "cubbyholes" (sets); because then we can study the relations between the sets and that is the beginning of the (very) elementary theory of sets. If G is the set of people who wear glasses (the "glass-wearers" of the world) and M is the set of males, we draw two (colored) strings like this:

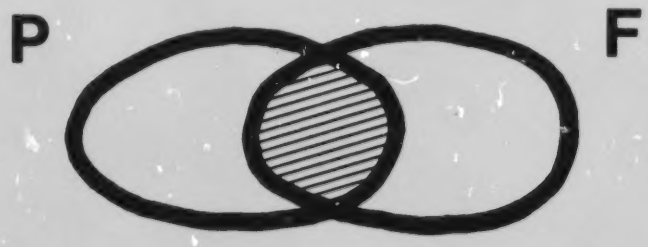
Now the classification scheme has to be concerned with two in/out decisions simultaneously. Consider Cleopatra—where does she go? She's not male and she's not a glass-wearer (glasses weren't invented then!) so she goes outside both strings.



On the other extreme, Ben Franklin was a male who wore glasses, so he has to go inside both the M-string and the G-string. Alexander the Great, by the same argument as the one for Cleopatra, was a man who didn't wear glasses so he goes in the M-string and outside the G-string. Eleanor Roosevelt did wear glasses and was not a male, so she goes in the place indicated above. Note that every object x in the world has its place in the string picture above; there are four possibilities:

1. $x \in G$ and $x \in M$
2. $x \notin G$ and $x \in M$
3. $x \in G$ and $x \notin M$
4. $x \notin G$ and $x \notin M$

Of course, there are situations where it is <u>impossible</u> for anything to go into one of the cubbyholes indicated above. For example, if P is the set of all U. S. Presidents, and F the set of all females, then we have (as of 1976) <u>no</u> x such that $x \in F$ and $x \in P$.



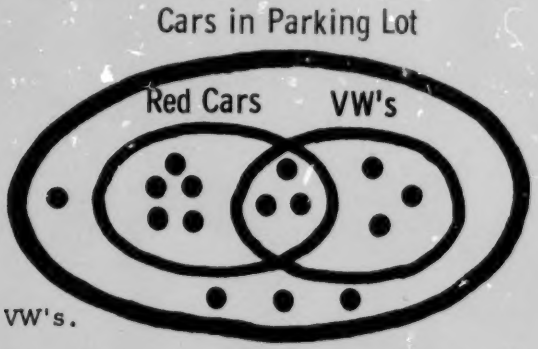
We can indicate this state of affairs by hatching the appropriate region—to suggest that no dots should be placed in the region in question.

Thus the language of strings and dots gives children a precise (and nonverbal) way of recording and communicating their thoughts about classifying things. The abilities to classify, to reason about classification, and to extract information from a classification scheme are important skills for everyday life, for intellectual activity in general, and for the pursuit and understanding of mathematics in particular. Ordinarily, people are not expected to deal with such problems until their command of their natural language is fully developed and until they face classification problems in adult life. It is not that an ability to classify requires special intelligence, but that it requires special modes of thinking. The unique quality of the language of strings is that it provides a nonverbal language that is particularly suited to the mode of thinking involved in classification, thus freeing young minds (in some cases as young as four years of age!) to think logically and creatively about classes of things and report their thinking, long before they have the verbal skills that were thought to be required.

In this strand we set up situations and ask carefully phrased questions that continue to develop skills in classification, always remembering that the skills grow out of such experiences but that an ability to draw strings and arrows is not an objective in itself; the mode of thinking involved is the objective. Thus it is extremely important for us to set up our situations carefully. The objects we ask children to classify must have well defined attributes; otherwise, the children's attention will be distracted by the problem of deciding whether a certain object does or does not have a certain attribute. For example, consider this situation. A teacher suggests that the blue string on the board is for all children in the class with dark hair and the red string is for all children in the class who are tall. She then invites a child, say John, to place a dot for himself on the diagram. He places a dot inside the blue string and outside the red string and labels it "J". There is a flurry of mixed agreement and disagreement in the classroom. John has brown hair and he thinks it is dark, but some say it is really light. John doesn't think he is tall, but someone insists that he is taller than most of the other children in the class. As a result, most of the energy in the class is directed toward making a decision about what attributes John has in this situation rather than how to classify John relative to those attributes. More care in choice of attributes would have avoided such confusion. For example, if the blue string were for all children in the class whose birthdays are in January and the red string for all the children who have a sister, John's choice of a dot for himself would be criticized strictly on its placement (i.e., his classification of himself) and not on his decision as to what attributes he has.

It is quite clear, then, why we make such extensive use of A-Blocks (attribute blocks) and sets of numbers in designing classification situations. Children can immediately say whether or not a block is red, whether or not it is a square, etc., and they know whether or not a given number of a set of numbers is less than 15, whether or not it is odd, and so on.

One reason for classifying objects is to be able to count the objects that have a certain attribute. For example, in a parking lot there are eight red cars and six VW's. If there are four cars in the lot that are neither red nor VW's, can we be sure there are 18 cars in the lot? Suppose we counted the cars and found there are exactly 15 in the lot. Is this possible? A string picture immediately settles this apparent paradox:



There are three red VW's.

Relations: The Language of Arrows

In almost everything we do or think about we are intensely interested in the relations between things. This is true in our everyday lives, in school, in our careers, and in all scientific pursuits—we are always trying to establish, explore, and understand relations. Consider the following simple example. A girl comes running up to her mother and shouts: "Mommy, Mommy, Fido bit the mailman!" The girl is reporting to her mother that a certain relation has suddenly been established between the family dog and a certain mailman. Fido bit the mailman. The existence of the "biting relation" between Fido and the mailman has many important practical consequences. From the mailman's point of view, it is presumably a painful one and may lead him to consult a physician. From the mother's point of view, it is at the very least embarrassing, and might even lead to a potential law suit; from Fido's point of view it will likely lead to punishment of some sort. For all these reasons

and many others, the girl feels it urgent for the mother to know of the existence of the "biting relation" between the dog and the mailman.

To show our keen interest in relations and the great variety of relations that can exist between things, consider the following short list of everyday life sentences. Each sentence asserts a relation between one object (person or thing) x and another y.

1. Johnny is taller than Mary. (Establishes a relation between Johnny and Mary in terms of their height.)
2. Johnny is Mary's brother. (Establishes a blood relation between the two.)
3. Johnny sent Mary a valentine.
4. Johnny borrowed $1 from Mary.
5. Johnny hates spinach. (A very common relation between a child and a vegetable.)
6. Tootsie Rolls are 2¢ cheaper at Store A than at Store B. (Establishes a relation between Store A and Store B in terms of their respective prices for Tootsie Rolls.)
7. Giving is more blessed than receiving. (This example indicates that there are relations among relations! A certain relation (giving) is more blessed than another relation (receiving).)

Consider also the following sentences taken from various school subjects. Each sentence talks about a certain relation.

English Literature:

1. Mark Twain is the author of "Tom Sawyer".

xx

2. Shakespeare lived before Hemingway.

3. "Huckleberry Finn" is a sequel to "Tom Sawyer".

Science:

1. Lead is heavier than aluminum.

2. Aardvarks eat ants.

3. Phobos is a moon of Mars.

4. Malaria responds to quinine. (A relation between a disease and a certain drug.)

History:

1. Brutus killed Julius Caesar.

2. Louis XIV was King of France. (A relation between a person, Louis XIV, and a country, France.)

3. Hitler hated Jews. (A relation between a person, Hitler, and a class of people, the Jews.)

4. President Kennedy was assassinated in Dallas. (A relation between a person and a city.)

Let's hope that the examples above have made our main point: we are interested in the relations between things. Mathematics is no different from everyday life, English literature, science, or history. To study mathematics means largely to study the relations between mathematical objects.

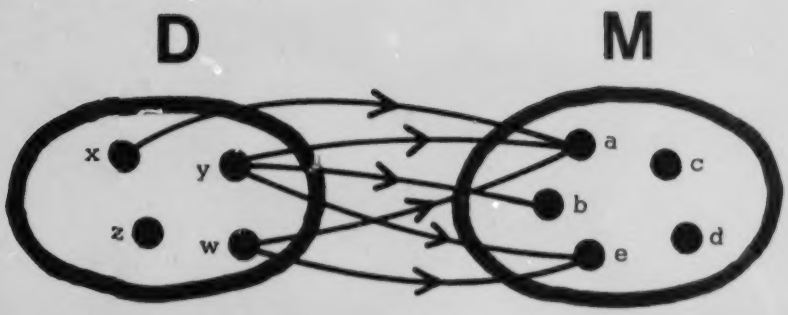
As always, if we are to study anything seriously, we will need a language, a way of symbolizing the things we want to study. A very apt language indeed for studying and talking about relations is the language of arrows. This is the language that is exploited systematically throughout this strand

for thinking and writing about relations. Let's quickly remind ourselves how the language of arrows works. We have already agreed earlier to represent objects by dots. In the story of Fido biting the mailman, we would represent both Fido (F) and the mailman (M) by dots:

What could now be more natural to represent the fact that Fido bit the mailman than by drawing an arrow from Fido to the mailman, like this:

In fact, with a little imagination the arrow suggests to our mind's eye quite vividly Fido's rushing towards the mailman and biting him. Note that, of course, we have drawn the arrow <u>from</u> Fido <u>to</u> the mailman. The reverse arrow (from the mailman to Fido) would mean the mailman bit Fido (in the spirit of the classical "man bites dog" saying!).

In general, if we are trying to picture any relation R between objects (biting, being the brother of, sending a valentine to, borrowing $1 from, and so on) we draw an arrow from object x to object y whenever x relates in the specified way to y. Suppose for example, D is a set of dogs and M a set of mailmen, and the red arrows represent the relation: bites.



Note how very vividly our arrow diagram presents the state of affairs. The answers to the following kinds of questions can be read off from the diagram almost instantly:

Which dog seems the most vicious?

Which dog seems the gentlest?

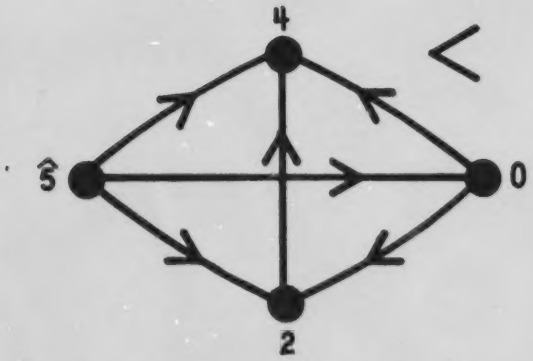
Which mailmen seem to get along best with the dogs?

Of course, frequently we are interested in comparing and contrasting two or more relations simultaneously. In that case we simply draw two superimposed arrow diagrams and code the arrows by color to show which arrows go with which relation.

In summary: Arrow diagrams are a very handy graphical representation of a relation, somewhat the same way that a blueprint is a very handy graphical representation of a house. By means of arrow diagrams we can represent important facts about a given relation or relations between objects in a very simple, suggestive, pictorial way—usually much more conveniently than the same information could be presented in words. The convenience of arrow diagrams has of course important pedagogical consequences for introducing children to the study of relations in the early grades. A child can "read"—and also draw—an arrow diagram of a relation long before he or she could read or

present the same information about that relation in words or other symbols. The situation here is the same as in so many other cases. The problem of presenting certain ideas to children lies not in their intellectual ability to grasp the ideas; rather, the limitations are often just mechanical, motor problems of writing or reading about the ideas in conventional systems. The arrow diagrams have all the virtues of a good notation: they present information in a clear, natural way; they are attractive, colorful things to look at; they are easy and fun for children to draw.

In the discussion above, we introduced and explained the arrow diagrams in terms of a relation that comes from everyday life: dog bites mailman. Obviously, in the mathematics class we are also interested in relations between mathematical objects like numbers, lines, triangles, outcomes of experiments, and so on. However, we showed how arrow diagrams can be used to picture nonmathematical relations just as well as mathematical ones, for an important reason—to show that the mathematics class is not an isolated world in which we play exotic games totally different from what we do in the rest of our lives. The important point here is simply this: thinking about and exploring the world of numbers is an activity very similar to thinking about and exploring any other world, e.g., the world of dogs and mailmen. Moreover, the tools we use to explore and understand the everyday world (e.g., arrow diagrams) are useful in exploring and understanding the world of mathematics. Conversely, the tools we develop and use to help us think about mathematical things are very likely to help us in thinking about other kinds of things also. To see the analogies, consider the following simple arrow picture, where a red arrow from x to y means x is smaller than y:



To make our point more graphic, we imagine the numbers (which are, after all, things) talking to each other via the red arrows and saying: I am smaller than you.*

Now we can ask such questions as:

1. Who does the most talking ? (i.e., which is the smallest number?)

2. Who does the most listening? (i.e., which is the largest number ?)

We leave it for the reader to find questions analogous to (1) and (2) above in the dogs and mailmen situation discussed previously.

One of the purposes of this strand is to exploit the power of the language of arrows to help children think logically about relations between things. Again it must be remembered that our goals concern the thinking process and not the mechanism. Thus we must resist making the ability to draw prescribed arrows an objective in itself, just as we must resist viewing an arrow diagram as just another format for drill problems in arithmetic.

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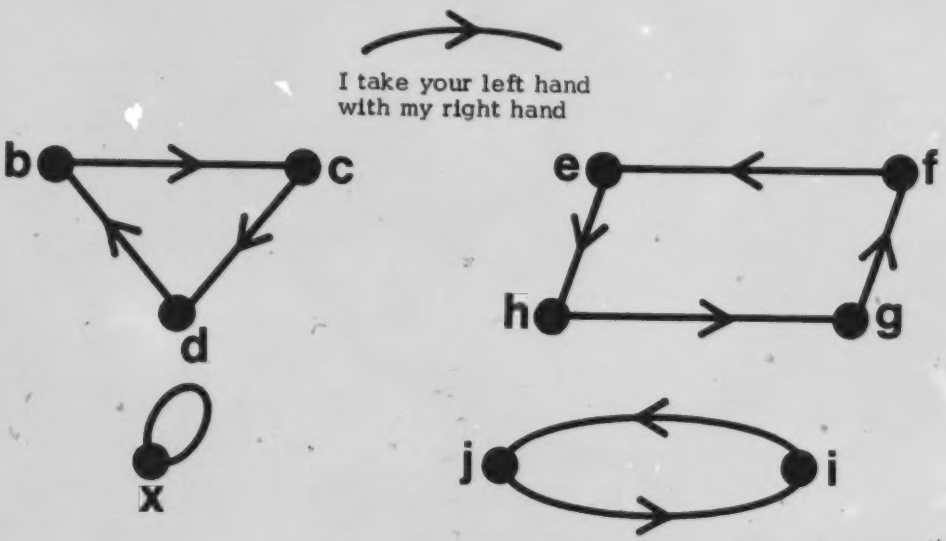
* Aesop, to make his points, used talking wolves and foxes and dogs. In the same spirit, we like to have the numbers talk. After all, the number 2 probably knows more about its relations to other numbers than anyone else, so why not have it tell us what it knows?

There is another educational bonus that occurs when arrow diagrams are used imaginatively. A story can be captured and described by an arrow diagram, and conversely, an arrow diagram can suggest many possible stories and games, allowing children to expand their creativity in many ways. In this sense, an arrow diagram resembles a script for a play or a choreographer's dance diagram. For example, suppose each child in the class is told to do the following: With your right hand take the left hand of one child. What confusion! When the instructions are eventually carried out they can easily make a record of this event (or it might even turn out to be a dance) with an arrow diagram that might look like this:



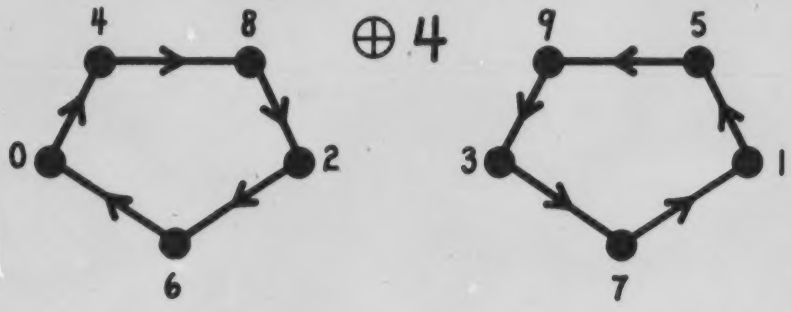
(Notice that student x took the easy way; he took his own left hand with his right hand.)

Conversely, given such an arrow diagram (which describes a relation called a permutation, because every dot has exactly one arrow from it and exactly one arrow to it), the children can use it as a "dance plan" and follow its directions in the next "dance". Of course, the same diagram might suggest

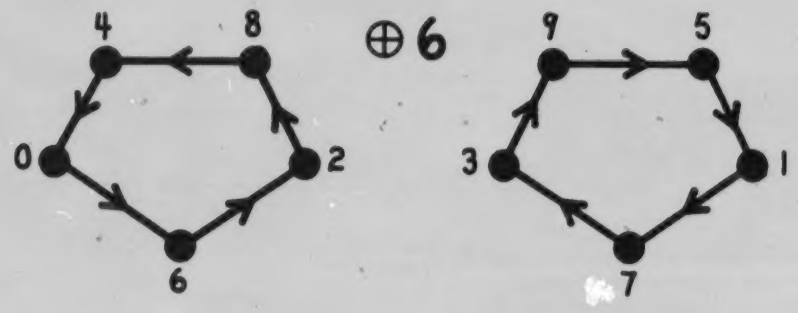
a wide variety of stories that children can tell, such as:

> "On Christmas Day, everyone in our big family gave exactly one gift to someone in the family and received exactly one gift from someone (not necessarily the same person who received his gift) in the family. Al (denoted "x" in the diagram) knew what he wanted, so he gave his gift to himself. Beth (b) gave a doll to Clara (c), ..."

It is not surprising that such arrow diagrams, and the games suggested by them, yield important mathematical concepts. Consider, for example, the following story situation in which the numbers 0, 1, 2, ..., 9 become separated from the rest of the whole numbers and are looking for some game to play. They decide they can't play their usual addition games, for that would require all the whole numbers to be present. So the ringleader, Zero, suggests a new "addition" game, $\oplus$, that they can play, in which $1 \oplus 7 = 8$, $3 \oplus 6 = 9$, $2 \oplus 8 = 0$, $7 \oplus 6 = 3$, etc. Now they can all play the game because they use only the last digit of the ordinary sum for this new kind of sum. "Let's try the $\oplus 4$ snake dance!" shout the numbers. And they do.



"Also, the ⊕6 snake dance!"



"The same dance, but in the opposite direction," they notice.

The arithmetic of the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} determined by this new operation, ⊕, illustrates some very basic concepts in mathematics, but at this level it is just one more interesting experience that generates good mathematical thinking.

The general aim of The Languages of Strings and Arrows strand is to suggest situations that are inherently interesting and thought provoking, and to give children modes of thinking and appropriate languages with which they can organize, classify, and analyze given situations.

By this time your students should be very familiar and quite comfortable with both the language of strings and the language of arrows. The content of this strand for the present semester builds in a variety of ways on this fluency with the languages. Each of the three chapters included in this strand presents situations quite distinct from those of the other two. Chapter 1 continues the work begun in The World of Numbers strand on the relation "is a divisor of", utilizing both the language of strings and the language of arrows;

then the String Game with Numbers is introduced to further support the concepts of multiple and divisor.

In Chapter 2, the verbal ability of your students is recognized. The language of strings is used in various situations which explicitly depend on the meaning of sentences such as: "At least four of the dogs are brown." "At most three of the dogs are brown." "Exactly two of the dogs are brown." "Both this dog and that dog are brown." The lessons of this chapter are by no means English lessons. They utilize the student's reasoning powers as well as the numerical knowledge he or she has.

In Chapter 3, the language of arrows is used in a completely nonverbal manner. The internal structure of the language is exploited to pose situations involving the composition of relations.

STRING GAMES

By this time your students should have developed a fairly comfortable feeling for multiples of a number, but they are just beginning to catch the idea of a divisor of a number. This chapter sets up various situations in which the relation "is a divisor of" is reviewed and used in a number of settings. The review follows more experiences with the relation "is a multiple of", and then the students are involved in thinking about divisors of numbers in string pictures, in arrow diagrams, and in playing several expanding versions of the String Game. By the end of the chapter they are playing the version of the String Game in which divisors of certain numbers, as well as multiples of certain numbers, are among the string cards. In the process, thinking about and dealing with divisors of numbers is becoming a more natural part of the students' experience.

CAPSULE LESSON SUMMARIES

Section 1.1:	Play the String Game with A-Blocks several times. This is a review and an extension to three strings and the "not" string cards. Review multiples of certain numbers in preparation for the String Game with Numbers.
Section 1.2:	Play the String Game with Numbers using two strings and with the string cards involving multiples of certain numbers and the order relation on numbers.
Section 1.3:	Review the notion of the positive divisors of a positive integer. Then extend the String Game with Numbers to include them.
Section 1.4:	Continue the experiences of Section 1.3 in order to gain more insight into the relation "is a divisor of".
Section 1.5:	Present more complex situations involving the relation "is a divisor of".

1.1 The String Game with A-Blocks

If your class began the CSMP curriculum in third grade or earlier, they will have played the simplest version of the String Game with A-Blocks some months ago. In this case, the following lesson will serve to review this simple version and to extend it to the version in which more attributes are included than before.

If, on the other hand, your class entered the CSMP curriculum in the fourth grade, they played this extended version of the String Game with A-Blocks quite recently. In this case, you should decide whether to play a few rounds of this game with them as a refresher and then go directly to Exercise 3 of this lesson, or to do only Exercise 3 and then regard the remainder of the period as part of a free day.

In either case, it is suggested that you refresh your memory about the rules of the game and the techniques for organizing it by reading the appendix to this chapter: The String Game with Shapes, Version A, which you will find at the end of this strand on pages 35 to 49. (If you are teaching the CSMP curriculum for the first time this term, the reading of the appendix will serve as your introduction to this fascinating game.)

Exercise 1:

T: Do you remember the String Game? We are going to play it again, this time with three strings.

Show the class the nine string cards (consisting of "Big", "Little", the three shapes, and the four colors as on page 39 of the appendix).

Separate the class into teams A and B and play Variation #2, as in Example 1 on page 41 of the appendix.

Exercise 2:

T: Now let us make the game a little more interesting. We will use more string cards than before.

Show the class the same nine string cards as before and also include the nine string cards on which is added the word "not" (shown on page 42 of the appendix).

T: For example, show me a piece that is not green.

Hold up the "not green" card. Accept several responses until it is clear that any red or blue or yellow piece of any size or shape will suffice.

T: Show me a piece that is not a triangle.

Hold up the "not Δ" card. Accept some responses until they seem to understand that any circle or square of any color or size will suffice. Then draw two strings on the board and play the game given in Example 1 of Version B: The String Game with Shapes, using "not" cards, as described on pages 42 to 44 in the appendix. If the game moves quickly, play another, say Example 2 or Example 3 of Version B on page 43.

Exercise 3:

In this part of the lesson, the purpose is to determine how much the students remember about multiples, both positive and negative, and then to review and reinforce the concept at the level where you find the class. In what follows go as far and as deep into the review as you feel necessary.

T: Can someone name a multiple of 3 ?

S: 9.

T: A larger one?

S: 23.

T: Explain to me why 23 is a multiple of 3 ?

In helping this student to decide whether 23 is a multiple of 3, you might ask him or her to draw a +3 snake from 0 to see if it hits 23.

S: I mean 24.

T: How many times 3 is 24 ?

S: Eight, because 8 x 3 = 24.

T: Can anyone name a larger multiple of 3 ?

Get students to explain some larger multiples, such as 63, by saying "21 x 3 = 63" or "63 ÷ 3 = 21". Continue with larger multiples of 3.

T: Are there any negative numbers that are multiples of 3 ? What if I look at this part of the +3 snake?

As you say this, extend the arrow picture on the board as follows:

Let the students tell you the numbers for these dots as you point to them: $^{-}3, ^{-}6, ^{-}9, \ldots$

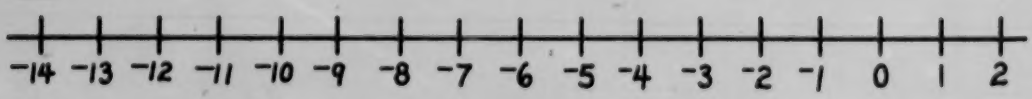
T: These negative numbers are also multiples of 3. Is 0 a multiple of 3?

S: Yes, it's on the +3 snake.

T: We decided that 63 is a multiple of 3. Is $^{-}63$ a multiple of 3?

S: Yes, because 63 is 21 jumps ahead, so $^{-}63$ is 21 jumps back.

Draw a portion of the number line:



T: Which is larger, $^{-}12$ or $^{-}3$?

S: $^{-}3$. I remember that a score of $^{-}3$ is better than $^{-}12$ in the basketball game.

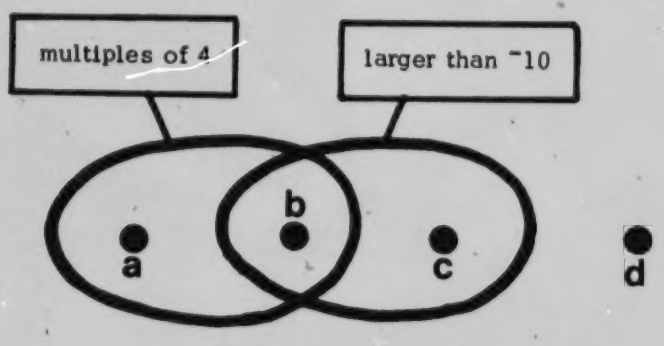
S: And because $^{-}3$ is to the right of $^{-}12$ on the line. (Or $^{-}3$ is closer to 0 on the number line.)*

* If two numbers are both negative, the one closer to 0 is the larger.

T: Is there a multiple of 3 which is larger than $^{-}8$ and smaller than $^{-}4$?

S: $^{-}6$. I can see it on the number line. That's the only one.

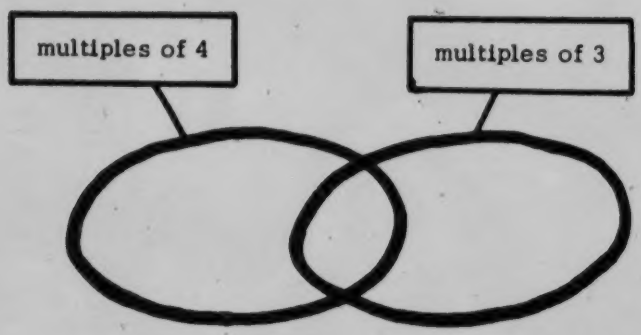
In the same way review multiples of 4 for a few minutes. Then draw this string picture on the board: (Do not label the dots. They are here just to make the description of the lesson easier to follow.)



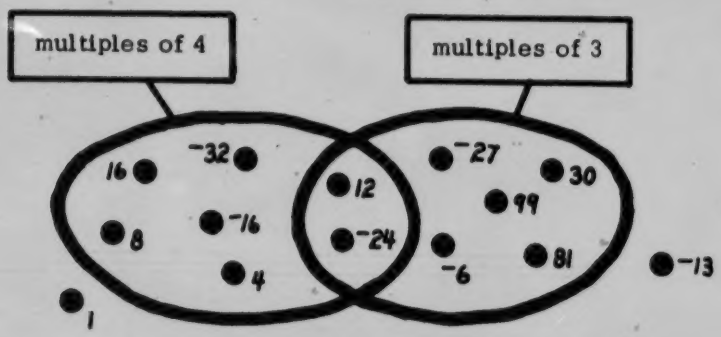
T: The red string is for all multiples of 4, and the blue string is for all numbers larger than $^{-}10$. What numbers could these dots represent?

As you point to each dot, call on students to give a number. For the dot labeled "b" accept any of $^{-}8$, $^{-}4$, 0, 4, 8, ... For the dot labeled "a" accept any of $^{-}12$, $^{-}16$, $^{-}20$, ... For the dot labeled "c": $^{-}9$, $^{-}7$, $^{-}6$, $^{-}5$, $^{-}3$, $^{-}2$, $^{-}1$, 1, 2, 3, 5, 6, 7, ... For the dot labeled "d": any not included above, i.e., $^{-}10$, $^{-}11$, $^{-}13$, $^{-}14$, $^{-}15$, $^{-}17$, ... (If a student gives an object which is not a number, accept that as well and reinforce the fact that "d" doesn't have to be a number.)

Use each response as an opportunity to review positive and negative multiples of 4 and the relations "larger than" and "smaller than" for positive and negative numbers. Also let students add dots to the picture and label them. If there is time remaining in the period, draw the following string picture and repeat the exercise:



After some questioning, your picture might look like this:



Here is an alternate way of presenting the material for the first part of Exercise 3 which utilizes hand-calculators. The instructions allow you to proceed with only two calculators, one for the teacher and one for a student. (If any of your students have their own calculators, they should be encouraged to bring them from home for this lesson.)

The student with the calculator should be in front of the class and answer any of the questions relating to the calculator when the display is exposed. Any of the children can answer when the display is hidden. All should be encouraged to think just as carefully and stay just as involved as if they had a calculator with a hidden display. The student with the calculator can check all the answers. For different parts of the lesson, choose a new student to work the calculator in order to give as many of them the experience as possible.

When giving instructions, always do so slowly and clearly.

T: Turn the calculator on.
Slowly press: [+], [3], [=], [=], [=], [=], and so on.
Read the numbers aloud that appear on the display.

S: 3, 6, 9, 12, 15, 18, 21, ...

T: (To the rest of the class) What is happening?

S: It's counting by 3.

S: All the numbers are multiples of 3.

Let the student continue reading the multiples of 3 which appear on the display for some time, perhaps until 99 appears.

T: Now clear the display and listen carefully to my instructions.
Press: [+], [3], [=], [=], [=].
What number is on the display?

S: 9.

T: Press again: [=], [=], [=], [=].
What number is on the display now?

S: 21.

T: Now keep silent until I ask you for a calculator check on the answers given by other students. Press: [=], [=], [=]. What number should be on the display?

S: 30

T: Calculator check.

S: That's right.

T: Press: [=], [=], [=], [=].

S: 42.

T: Check.

S: He's right.

T: Press: [=], [=]. What two numbers should have appeared?

S: 45 and 48.

T: Check.

S: That's right.

Let several students whisper in your ear the number of times they think [=] should be pressed to get from 48 to 60. Ask one student to respond orally. (Four times.)

T: Press: [=], [=], [=], [=], [=]. Who can name the numbers the two of us (the teacher and the student presently working the calculator) saw appear on the display?

S: 63, 66, 69, 72, 75.

S: She's right.

T: How do you think we could get back to 0 from 75 in such a way that we would see each number reappear in reverse order? (75, 72, 69, ...)

S: Press: [-], [3], [=], [=], [=], ... and so on.

T: All right, we'll do that and tell you when we get to 0. Now we are at 0. If we keep pressing [=], [=], [=], [=], what are the first four numbers that will appear?

S: $^{-}3$, $^{-}6$, $^{-}9$, $^{-}12$.

T: Will $^{-}25$ appear?

S: Yes.

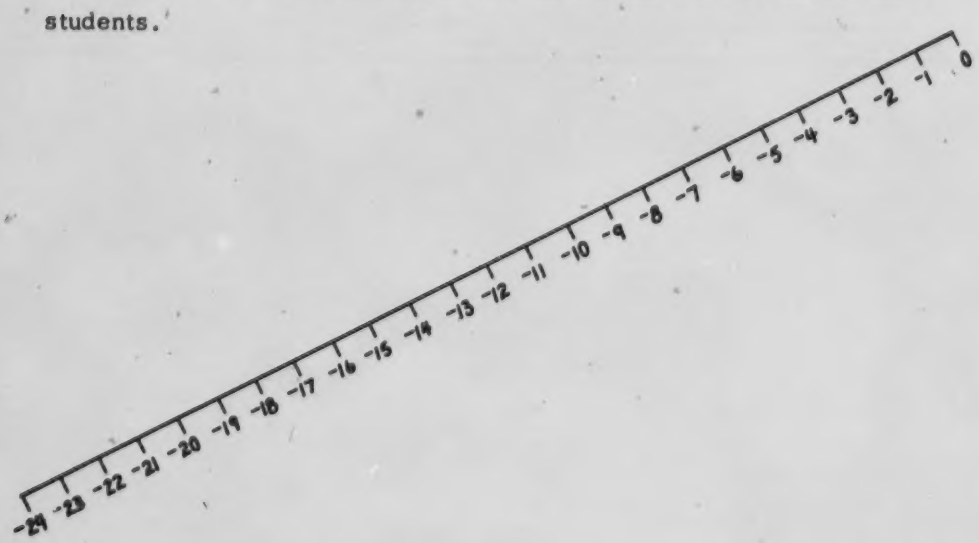
T: Explain how that will happen.

S: $^{-}15$, $^{-}18$, $^{-}21$, $^{-}24$ —Oh, no! I was wrong.

T: Which number is larger, $^{-}24$ or $^{-}12$?

S: $^{-}12$ because it's to the right of $^{-}24$. (Or because $^{-}12$ is closer to 0 than $^{-}24$ and they're both negative.)

Draw a portion of the number line to illustrate this clearly for all the students.



T: What about $^{-}60$?

S: Yes, because 20 x $^{-}3$ will be $^{-}60$.

T: So, $^{-}60$ is a multiple of 3 and 60 is a multiple of 3.
What are the multiples of 3 between $^{-}10$ and 0 ?

S: $^{-}9$, $^{-}6$, and $^{-}3$.

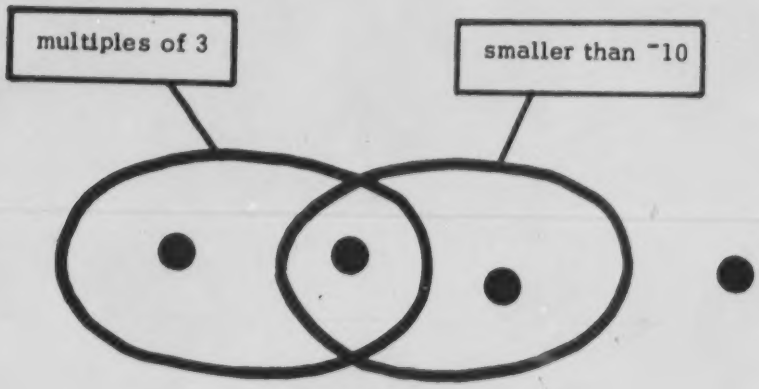
Have the student illustrate this on the number line. Continue Exercise 3 by reviewing multiples of 4 in a similar way for a few minutes and then by drawing the string picture on page 6 and proceeding as indicated by the dialogue following the picture.

1.2 The String Game with Numbers: Multiples and Order

Exercise 1:

Review the last part of Section 1.1 as follows.

Draw this string picture on the board:



T: The red string is for multiples of 3, and the blue string is for numbers smaller than ⁻10. What numbers could the dots represent?

As students suggest numbers for the dots, ask questions such as "Why?" or "Can you explain?" and then label the dots as they suggest.

T: Who can put a dot on the board for 0? for 7? for ⁻10? for ⁻16? for ⁻21? for 81? (Ask about numbers which have not yet been placed.)

As they place dots, you may need to reinforce the fact that ⁻21 is smaller than ⁻10, ⁻7 is larger than ⁻10, etc., by drawing a part of a number line to reinforce the idea visually. Ask for more dots to be placed until each region of the picture has several dots in it.

Exercise 2:

T: We are going to play the String Game again, but this time with numbers.

Show them the string cards and number cards for the new game (see page 44 of the appendix) and explain each of the string cards. Then erase the board, draw a two-string picture, and play at least two games. You can choose from among Examples 1, 2, 3 of Variation #1 of Version C: The String Game with Numbers: Multiples and Order, as described on pages 44 and 45 of the appendix, or you can make your own choices of the two string cards.

1.3 The String Game with Numbers: Multiples, Divisors, and Order #1

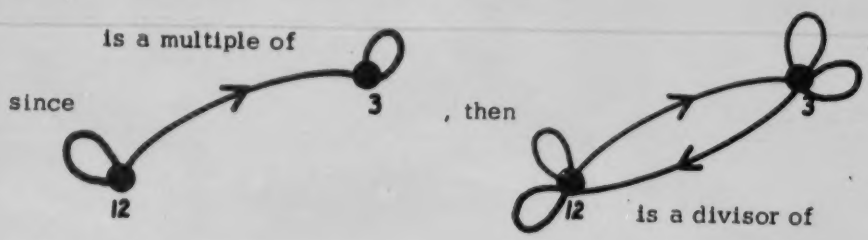
Allow about half the period for Exercises 1 and 2.

Exercise 1:

T: What are the positive divisors of 12?

S: 1, 2, 3, 4, 6, 12.

If necessary, review the meaning of the divisors of a number. For example, Why is 3 a divisor of 12? Because 12 is a multiple of 3, or because $12 \div 3 = 4$, or because four 3's are 12, or



T: What are the positive divisors of 20?

S: 1, 2, 4, 5, 10, 20.

T: What are the positive divisors of 10?

S: 1, 2, 5, 10.

Ask the same questions concerning divisors of other numbers such as 13, 6, 1, 21, 32, 120, etc. (Divisors of 120: 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120.)

T: Give me some numbers with exactly four positive divisors.

S: 6, because the positive divisors of 6 are 1, 2, 3, 6.

S: 8, because the positive divisors of 8 are 1, 2, 4, 8.

S: 14, its divisors are 1, 2, 7, 14.

S: 21. The positive divisors of 21 are 1, 3, 7, 21.

T: Can you give me some numbers with exactly six positive divisors?

S: 24. Its divisors are 1, 2, 3, 6, 12, 24.

T: Isn't 4 a divisor of 24? (Yes) How about 8? (Yes)

S: 18. Its divisors are 1, 2, 3, 6, 9, 18.

S: 28. Its divisors are 1, 2, 4, 7, 14, 28.

Accept any others that the class agrees on.

T: Give me some numbers with exactly two positive divisors.

S: 7. Its positive divisors are 1 and 7.

S: 3.

Other such numbers are: 2, 5, 11, 13, 17, 19, 23, 29, and so on.

S: 51.

S: No good. Three is a divisor of 51 and so is 17.

T: What do we call the numbers with exactly two positive divisors?

S: Primes.

Write on the board: Positive Primes. Then ask for the smallest positive prime, the next, the next, etc., and with each response add to the list.

Positive Primes: 2, 3, 5, 7, 11, ...

Keep this going until interest in the activity lags.

T: Isn't 1 a prime?

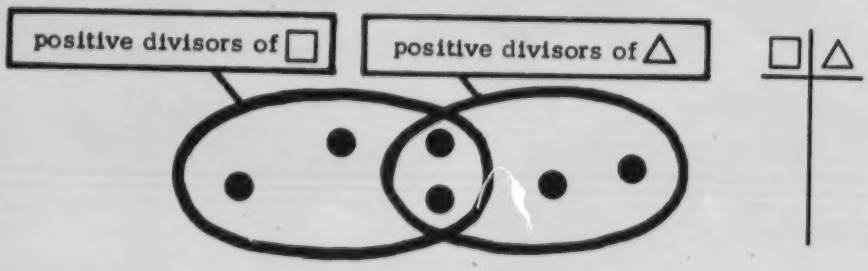
S: No. 1 has only one positive divisor.

T: What is its only positive divisor?

S: Itself (i.e., 1).

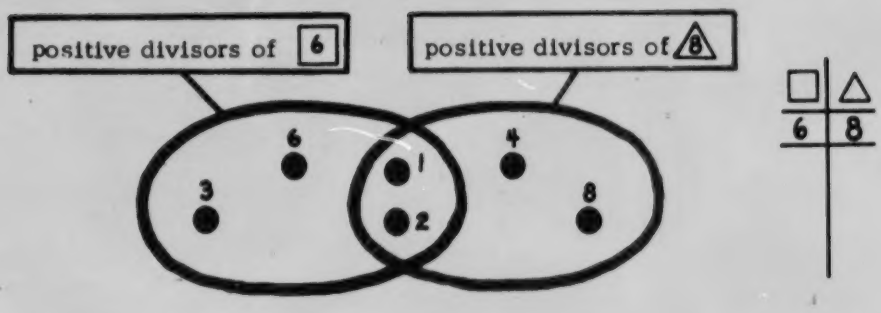
Exercise 2:

Draw the following string picture and table on the board:



T: Do you remember this game? The red string is for all the positive divisors of the number in the box (exactly four of them) and the blue string is for all the positive divisors of the number in the triangle (There are again, exactly four of them). What could the numbers in the box and in the triangle be?

Invite students to write their numbers in the box and the triangle and label the dots. For example, one solution is:



Label the dots according to this solution and put "6" and "8" in the proper columns of the table. Then erase "6" and "8" and the labels of the dots in the string picture (but not in the table).

T: What other numbers could be in the box and triangle?

S: 8 in the box and 6 in the triangle.

Accept this solution with a shrug, remarking that it certainly must be all right since there are exactly four divisors of each.

T: Another? This time two new numbers.

S: 14 and 22. The divisors of 14 are 1, 2, 7, 14 and the divisors of 22 are 1, 2, 11, 22.

S: 10 and 26.

After each response, let the student label the dots and put the numbers in the table. After about three such responses that are confirmed by class discussion, ask the students to copy the string picture and the table and find more numbers for the table if they can. For some, this will be a matter of hit or miss experimentation, and some might not find any new pairs of numbers. For others, there may be some strategies forming in their minds.

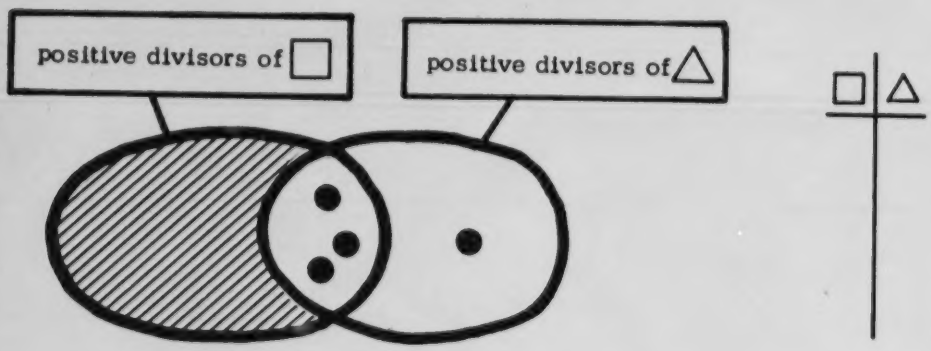
For example, some might begin to try two multiples of 2 (since all multiples of 2 have 1 and 2 as divisors) or two multiples of 3 (since all multiples of 3 have 1 and 3 as divisors) or multiples of 5 ... But do not suggest <u>any</u> sort of strategy, and allow any kind of experimentation.

Among the many possible pairs are:

□	△
6	8
10	26
8	14
14	22
26	34
6	27
15	21

□	△
33	39
10	15
35	55
14	21
35	77
22	33
77	35

Finally, erase what is on the board and draw this:



T: Suppose I put 4 in the box and 6 in the triangle. Would that be possible?

Direct the discussion to the fact that 4 does have three divisors and 6 does have four divisors, but there is a divisor of 4 that is not a divisor of 6. (4 is a divisor of 4 but 4 is not a divisor of 6.) So there would have to be a dot in the space that is hatched. (You might need to review the meaning of the hatching.)

T: There are exactly three dots inside the red string and exactly four dots inside the blue string. Is every dot in the red string also in the blue string?

S: Yes, because there are no dots in the red string that are not already in the blue string.

T: Is every dot in the blue string also in the red string?

S: No. One dot is in the blue string but not in the red string.

T: Who can give numbers for the box and the triangle?

S: 4 and 8.

S: I know another: 8 and 4.

S: That can't work this time. We can't switch them around.

Ask a student making a correct response to come to the board, label the dots, and place numbers in the box and triangle and in the table. After discussion has confirmed the answer, erase the labeling on the string diagram.

T: Copy the string picture and the table on your papers. Then see how many different pairs of numbers you can put in the table.

As you walk around the room checking the work, you can quickly spot gross

errors by looking to see whether the number in the box is a square number (such as 4, 9, 16, 25, ...) and the number in the triangle is a cubic number (such as 8, 27, 64, 125, ...). If not, the answer must be wrong. If so, check further to see whether the pair is correct. Some of the many possible solutions are given below:

□	△
4	8
9	27
25	125
36	216

This activity also should be allowed to progress by trial and error, even if no new solutions are found by some students. The experience is important even when a correct answer does not result. It is not expected that any students will see the strategy (namely that in "□" we must put the square of a prime and in "△" the cube of that same prime), but it could happen.

Exercise 3:

In the last half of the period, play the extended version of the String Game that is known as Version D: The String Game with Numbers: Multiples, Divisors, and Order as described on pages 47 to 49 in the appendix. For this experience, play with only two strings and pick two string cards at least one of which is a divisor card; for example, "Positive divisors of 5" and "Smaller than 50". To give a quick start, place two pieces from each team's stack on the diagram.

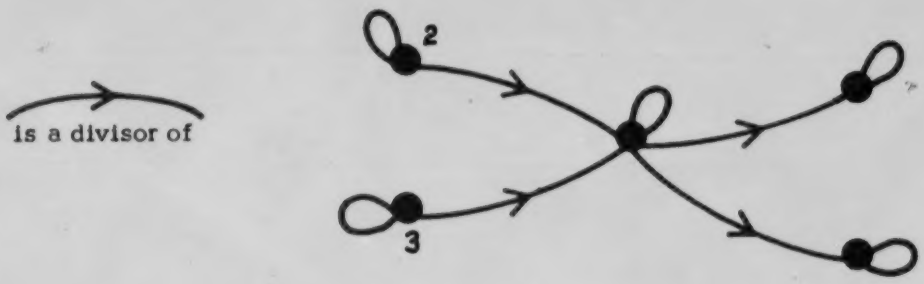
If there is more time, play a second round, this time choosing two divisor string cards.

1.4 The String Game with Numbers: Multiples, Divisors, and Order #2

In this lesson the first two exercises are designed to continue the experiences with divisors. The increasing richness of these situations offer more possibilities for insights. Leave enough time at the end of the period for the third exercise which will consist in again playing the String Game with Numbers.

Exercise 1:

Draw the following arrow picture on the board:



T: There are exactly five positive numbers here. Two of them you already know. Can you find numbers for the rest of the dots? Remember: one dot—one object.

S: Let's put 4 in the middle.

S: No, 3 is not a divisor of 4.

S: Let's put 6 in the middle.

S: or 12!

T: Would 6 work? Is 3 a divisor of 6? (Approval from the class.)

Would 12 work? Are 2 and 3 both divisors of 12? (Approval.) Let's pick 6. (Write "6" under the center dot.) Now what could the other dots be for?

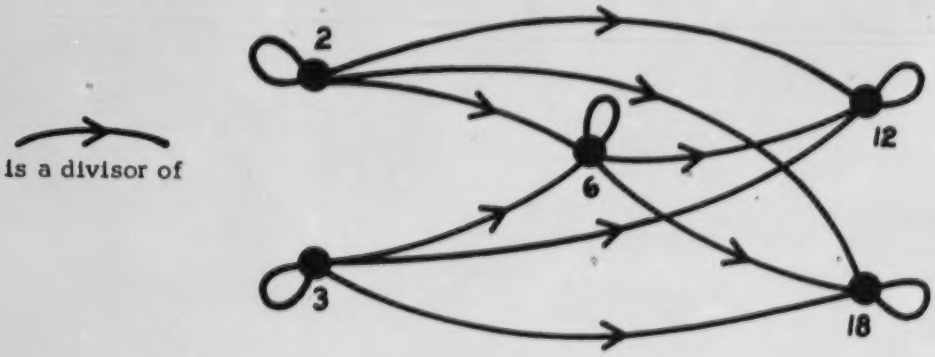
S: The top one could be for 12.

S: The bottom one could be for 18.

Accept any responses that are correct, asking for reasons ($12 = 2 \times 6$, $18 = 3 \times 6$, $24 = 4 \times 6$, etc.) Some students might say that the dots could be for any two multiples of 6 that are different, but do not impose this idea. Choose some correct responses, say 12 and 18, and write them on the diagram.

T: But I believe now we need more blue arrows. Can someone draw them?

There will be a flurry to draw them. Accept all efforts, letting students correct those that are not applicable, until the picture looks like this (the numbers you use might be different from these):



T: Do you think we could choose the last three numbers differently so that we wouldn't need more blue arrows?

Some students might suggest starting again with the original arrow picture and trying different numbers.

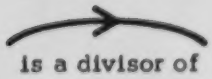
S: Yes, let's try new numbers.

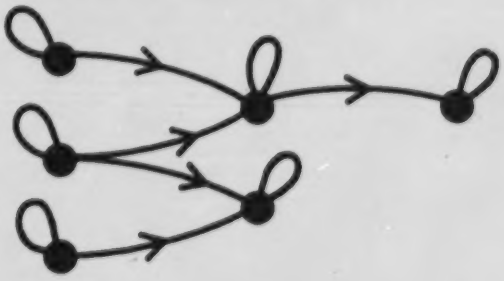
Others might believe there will always be new arrows needed.

S: No, I don't think so.

Don't spend much time on this. Be noncommittal as you listen. Of course, if some students give a reasonable explanation of why there will always be new arrows, you should praise them, but it is highly unlikely that you will get a good explanation. (The explanation is that the middle dot must be for a multiple of 6 and then the two dots on the right must be for multiples of 6, but every multiple of 6 is a multiple of 2 and also a multiple of 3.)

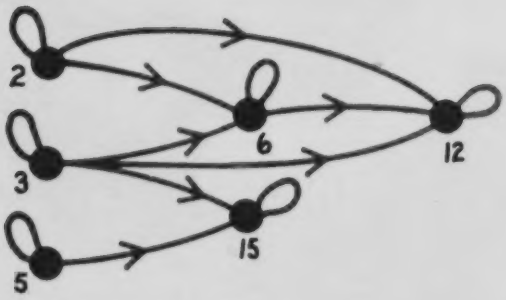
If the students enjoy this sort of experience, let them copy the following arrow picture, label the dots, and draw all missing blue arrows. Ask them to label the dots so that we get the least number of new arrows.



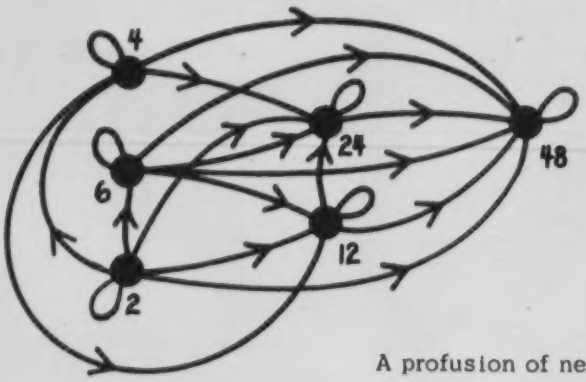


This problem has many solutions. Some students will start as in the preceding experience with 2 and 3 for two of the dots on the left, others will start with other numbers. In any case, the minimum number of new blue arrows will be two as in the following case:

is a divisor of



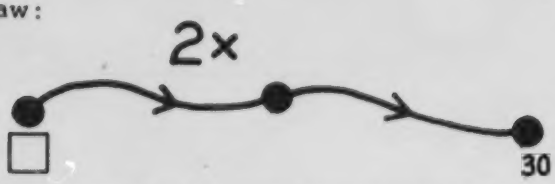
Other answers with more than two new blue arrows could look like this:



A profusion of new arrows!

Exercise 2:

Draw:



is a divisor of

T: Copy this picture on your paper and find as many numbers for the box as you can.

S: 15.

T: Will 15 work?

S: No, 30 would be for the second dot and we already have 30 for the third dot.

T: Yes. It was a good idea, but remember: only one dot for each number

S: $\frac{1}{2}$ or 0.5.

T: What is 2 x 0.5 ?

S: 1. It's okay, because 1 is a divisor of 30.

S: 1.

T: Will that work?

S: Yes, the second dot will be for 2, a divisor of 30.

S: 1.5.

T: Is that a solution?

S: Yes, the middle dot will be for 3, and it's a divisor of 30.

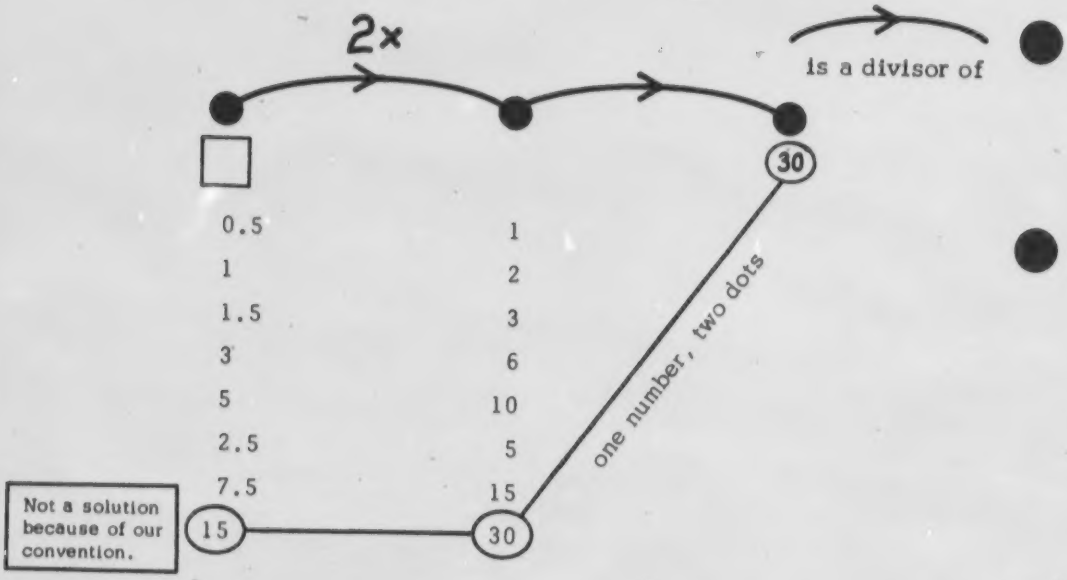
S: 3. Then 6 is in the middle and 6 is a divisor of 30.

S: 5. That puts 10 in the middle and 10 is a divisor of 30.

S: 2.5. Then 5 is in the middle. Okay.

S: 7.5. Then 15 is in the middle. Okay.

As each of these are suggested and explained, record them under the diagram. This is an example of what the board might look like (if there are still some divisors of 30 not in the middle column, keep asking for more until the class is satisfied that they have them all):



Exercise 3:

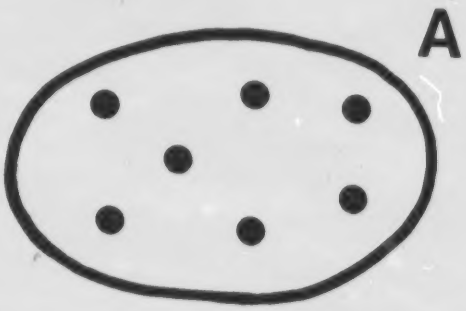
Try to reserve enough time at the end of the period to play a round or two of the String Game with Numbers, Version D (with multiples, divisors and order). It is probably best to use only two strings—unless you feel your class is ready for three.

1.5 The String Game with Numbers: Multiples, Divisors, and Order #3

This lesson consists of three exercises. The first two deal with divisors in settings which require some logical thinking and the third is devoted to the String Game with Numbers.

Exercise 1:

Draw on the board:



Write below the picture:

> A is a string for exactly seven positive integers and all these integers are less than 10.

T: I am going to write some statements about these numbers. You tell me whether they are true or false.

The following are the statements you will write. After you write each one, stop and let pupils argue about whether it is true or false. (The answers in parentheses are not to be written on the board, of course.)

(False)	1.	All the multiples of 6 are in A.
(True)	2.	At least one divisor of 6 is in A.
(True)	3.	At most four divisors of 6 are in A.
(True)	4.	At most one multiple of 6 is in A.

(True) 5. At least two divisors of 6 are in A.

(False) 6. At least one multiple of 12 is in A.

(True) 7. At most five divisors of 12 are in A.

As the discussion progresses, you might establish, through questioning, that there are exactly four positive divisors of 6 and all of these are less than 10. Also, there are exactly six positive divisors of 12, but only five of these are less than 10. Furthermore, the smallest positive multiple of 6 is 6, and this is the only multiple of 6 in A. But the smallest positive multiple of 12 is 12, so there are no multiples of 12 in A.

The reasons for the answers are as follows:

1. False, because there is a multiple of 6, say 12, that is greater than 10.

2. True. If there were no divisors of 6 in A, then there would be no dots for four of the nine integers from 1 to 9. But there are dots for seven of the integers from 1 to 9 so there must be at least one divisor of 6 in A. For this statement it is best to invite a student to try to label the seven dots with positive integers between 1 and 9 (inclusive) without naming a divisor of 6. He can't do it because he has too many dots.

3. True. "At most four" means "4 or 3 or 2 or 1 or 0". Since 6 has exactly four divisors, there can be no more than four divisors in A.

4. True. 6 is the only possible multiple of 6 in A.

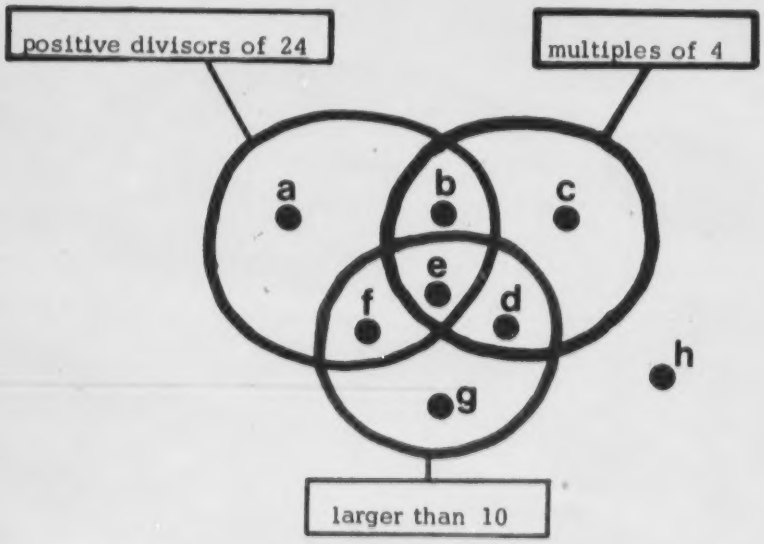
5. True. The same attempt as in statement 2 will show that we must name at least two of the dots with divisors of 6.

6. False, because the smallest positive multiple of 12 is 12 which is not in A.

7. True. 12 has five divisors between 1 and 9 (inclusive). Hence, there can be no more than five of its divisors in A.

Exercise 2:

Draw the following (but do not include the letters):

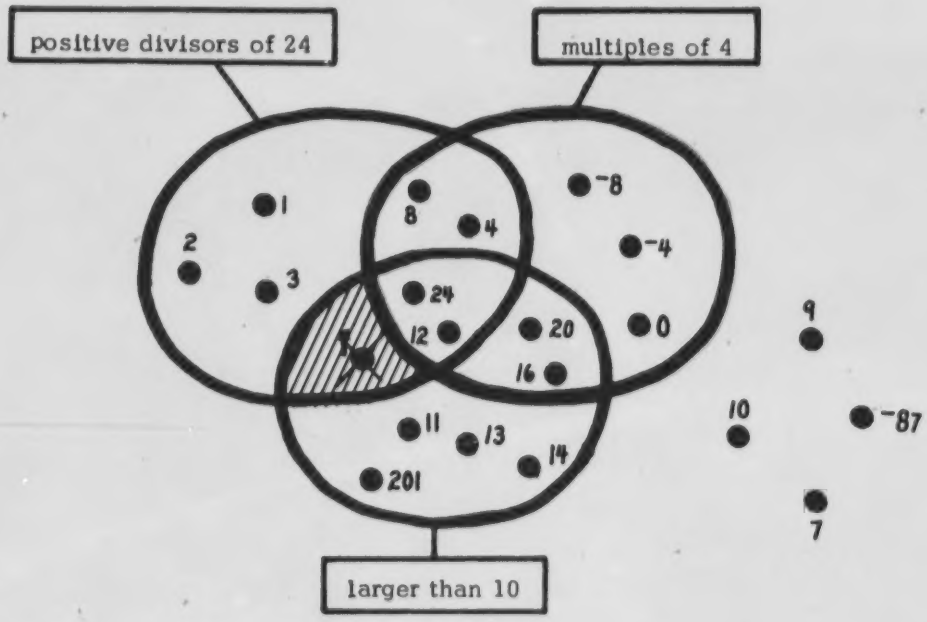


T: I've placed some dots in the strings. What numbers could they be for? When you give a number, tell why it belongs there.

You will need to require each response to be accompanied by statements about each of the three strings. For example, suppose a student labels the dot marked "d" with "12". He should say "12 is a multiple of 4—in the green string. 12 is larger than 10—in the blue string. 12 is a divisor of 24—in the red string. Oh, I see I should let "e" be 12, not "d", because 12 is in all three strings."

Notice that the region containing the dot marked "f" is empty and should be hatched because every divisor of 24 that is larger than 10 is a multiple of 4,

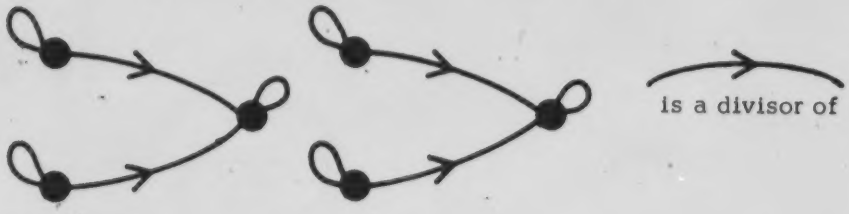
so either erase the dot marked "f" or cross it out when this is discovered. Continue in this manner until each dot is labeled correctly. If there is interest in this, let the students come to the board one-by-one and place a dot and label it. It might look like this:



Exercise 3:

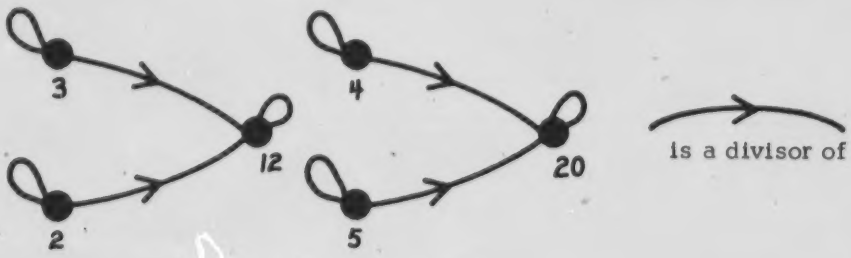
In the remaining portion of the period you can choose one of two alternatives. Either play the String Game with Numbers, Version D (by this time your class may be ready for the three-string game) or use the following activity, if your students enjoy arrow pictures involving the relation "is a divisor of".
If you choose not to use this activity at this time, you might wish to do it during a free day or whenever you have an additional 20-25 minutes. We also suggest that you play the String Game whenever you can arrange to do so, not just when it's scheduled in the lessons.

T: Now I will put a new picture on the board for you to copy and see what solutions you can find.



T: Listen carefully to the problem. Label the dots so that the picture is exactly correct, with just these loops and these arrows. When you have labeled the dots it should not be possible to add any other arrows. Remember, since all the dots are in the same picture you can't have the same number for two dots. When you think you have a solution, you can go to the board and put it there. Then we will all discuss whether your solution is correct.

One student might label the dots in this way:

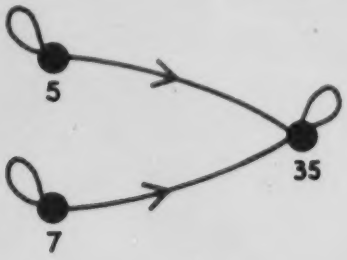


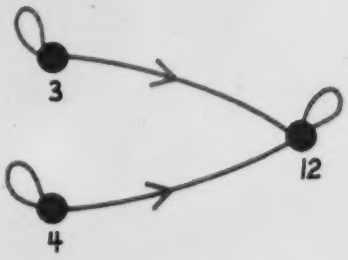
S: It's incorrect because it's not complete. 4 is a divisor of 12. We need an arrow from 4 to 12. Also there should be an arrow from 2 to 20 and one from 2 to 4.

S: Do all the dots have to be for whole numbers?

T: Yes, and positive also.

Another student might try this labeling:





T: Are 5 and 7 divisors of 35?

S: Yes.

T: Are 3 and 4 divisors of 12?

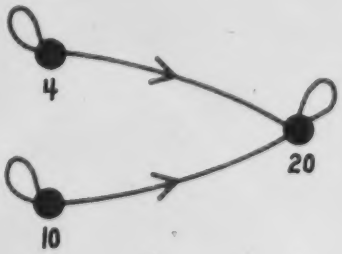
S: Yes

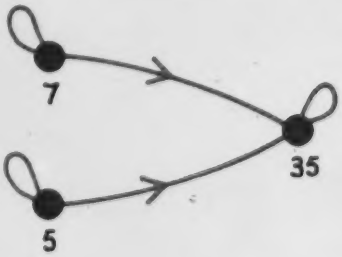
T: Can we draw any other arrows?

S: No.

T: Then Jenni's solution is a good one.

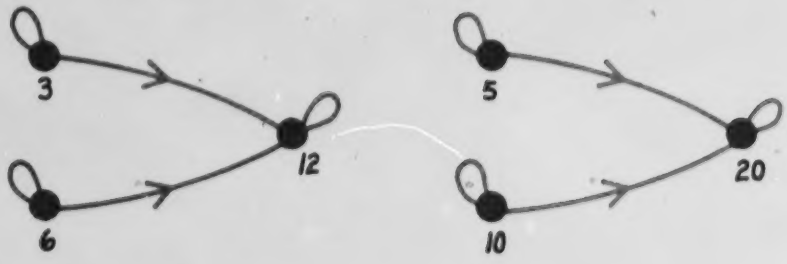
Still another student might suggest this:





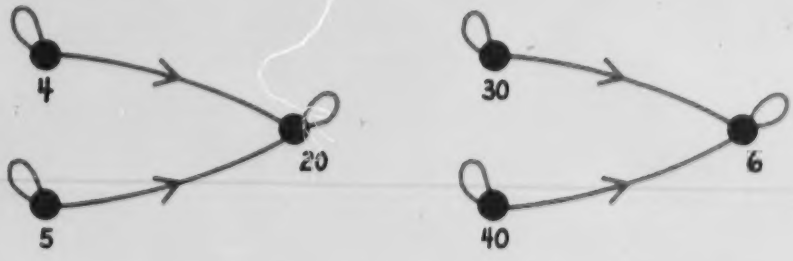
S: This won't work. You can draw an arrow from 5 to 20 and one from 5 to 10.

Another suggestion:



S: This is wrong. We can add an arrow from 5 to 10 and one from 3 to 6.

Still another:

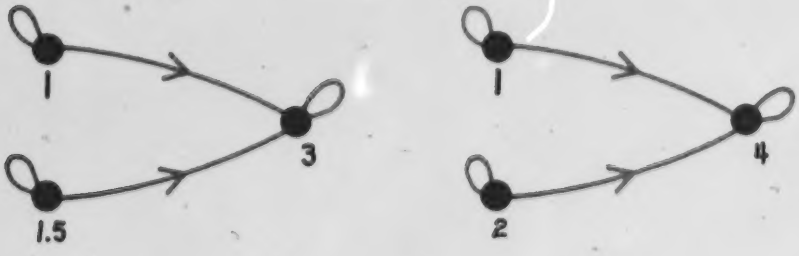


S: This is wrong. Arrows from 5 to 30, 5 to 40, 4 to 40, and 20 to 40 are missing.

T: Isn't there another reason?

S: 30 and 40 are not divisors of 6; 6 is too small. Six would have to be larger than 30 and 40.

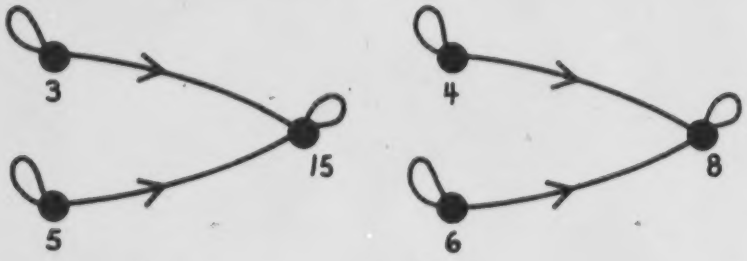
Another:



T: We can only use positive integers, remember.

S: Also there are two dots labeled "1".

Another student might suggest this:

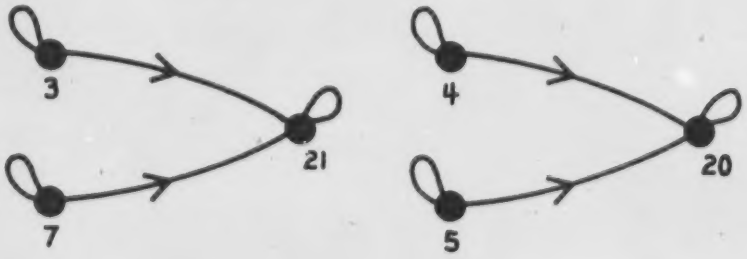


S: 6 isn't a divisor of 8.

T: Can you fix it?

S: (Crosses out the "8" and writes "12") Now we can add two arrows, 3 to 6 and 3 to 12, so it's still wrong.

Here is another suggestion:



T: Will this work?

S: No, 3 is a divisor of 20.

S: No it isn't! This one is O.K.

APPENDIX: The String Game

The String Game is introduced in Part I (Grade Two) of the CSMP Upper Primary Grades Program and used again in Part II (Grade Three). It gives children an opportunity, in a game-like atmosphere, to become familiar with the language of strings, while at the same time involving them in the kind of reasoning that will be developed and reinforced in various contexts throughout the program.

The String Game is first played using A-Blocks, but later on variations which make use of other objects (such as numbers) will make their appearance. In this appendix we provide you with the necessary information and examples to enable you to play the game using A-Blocks, and we begin the process of detailing the various possibilities open to you when you play it using numbers. The content of this appendix will be added to in later years of the CSMP Elementary School Curriculum as the children are introduced to new ideas. The examples given in this appendix are referred to in several of the lessons in this strand. However, we suggest that you do not restrict your playing of the String Game just to the few lessons in this semester when it has been scheduled. It has been found that it is most beneficial (and enjoyable) for children if you make a regular practice of playing the game about once every two weeks, or whenever you have an extra 10-15 minutes during the course of the day.

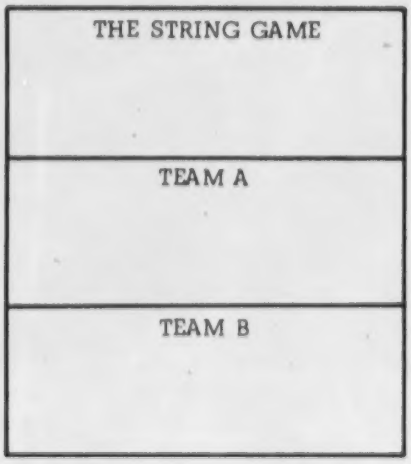
Equipment

The most easily manageable version of this game depends upon your having a magnetic chalkboard available. Many of the permanently mounted chalkboards in classrooms are magnetic, and you should test yours using one of the demonstration Minicomputer checkers. If your permanent chalkboards

are not magnetic, try any portable chalkboards (grid boards, etc.) that the school has available. In the case that you can locate a magnetic chalkboard, you will also need to obtain a fairly large sheet of metal (minimum size: 50 cm x 80 cm) for use as your Game Board. Your game pieces can be magnetized by gluing to the back of each one a small piece of magnetized material (available in many hobby and notions departments of stores). In the case of those string cards which are to be used face down only, glue the piece of magnetized material to the front of the cards, taking care not to obscure what is written thereon.

If you do not have a magnetized chalkboard available, you can use your regular chalkboard. In this case, your Game Board should be a large piece of poster board (minimum size: 50 cm x 80 cm) and the game pieces and string cards are attached by means of loops of masking tape stuck to the backs of the game pieces and the fronts of those string cards which are to be used face down. With this type of equipment, you should prepare yourself for running repairs by having sufficient masking tape available so that if some particular loop loses its stickiness, it can be replaced on the spot.

The Game Board should be marked out into regions as follows:



These markings should be made in paint (in the case of a metal Game Board) or using a heavy felt-tip pen (in the case of a poster board Game Board).

The game pieces (other than A-Blocks) and string cards should be cut out of sturdy cardboard, should be no smaller than 5 cm x 5 cm, and whatever writing is on them should be done with a heavy felt-tip pen. You will need two sets of string cards for each version of the game. One set is to be permanently attached to the upper section of the Game Board; the other set is referred to below as the "face-down set" and is to be used for labeling the strings that feature in the game. If you are using tape loops, you can avoid your face-down pieces sticking to each other when they are stored between games by only attaching tape loops to those two or three string cards that you are going to use in a given game and removing those loops when the game is over. Magnetized pieces, on the other hand, will cause no difficulty if they are permanently magnetized. The face-down set of string cards may be stored between games in a suitably sized manila envelope; the game pieces may be stored simply by attaching them to the Game Board.

Preparation for the Game

Draw two (or three, depending on which variation you are using) large, overlapping strings on the board using two (or three) different colors. Next to each of these strings attach one string card from the face-down set. Stand the Game Board on the chalk tray. Share the game pieces between the two sections of the Game Board labeled "TEAM A" and "TEAM B" by randomly placing one-half of the pieces in one section and the remainder in the other section. Divide the class into two teams using whatever method is acceptable to your class; e.g., boys vs. girls; tables A, B, C vs. tables D, E, F; etc. Call one team "Team A" and the other "Team B". Before any child takes a turn, correctly place an equal number (at least one) of each team's game pieces on the string picture. This removes from the first players

the necessity of operating on the basis of pure guesswork. You can influence how long the game will take by placing a smaller or a larger number of pieces to begin with.

The Rules of the Game

1) The game is played in silence. This is to avoid one child telling a teammate which piece to place where, or what the strings are for. Each child should have the opportunity of figuring these things out on an individual basis. Infringement of this rule by an individual is penalized by that individual's team losing its next turn.

2) The teams take turns.

3) The members of each team take turns coming to the board when it is their team's turn. They are to take one of the game pieces from their team's section of the Game Board and place it in one of the regions of the string picture.

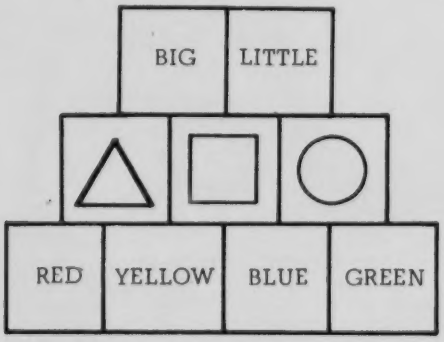
4) You are the judge. If the piece is correctly placed, you say "yes", it remains on the string picture and the child immediately has a second turn (no child may have more than two turns at any one trip to the board). If it is incorrectly placed, you say "no", and the child must replace the piece in question in his or her team's section of the Game Board and return to his or her seat. Obviously, it is of paramount importance that the judging be flawless. It might be advisable to prepare a "crib sheet" for yourself before each game showing the correct position of each game piece. If at any time you discover you have made an error, announce this fact immediately and rectify the mistake, either by moving an incorrectly placed piece whose position you had approved to its correct place, or by placing on the string

picture a correctly placed piece that had been returned to the Game Board because you had disapproved its position.

5) As soon as a team has correctly placed all of its pieces, the person who places the last piece may thereupon state what he or she thinks is written on each of the string cards. If these are all correct, the team has won. If a mistake is made (even if it's only in the case of one of the string cards), simply indicate that the given identification is not correct (do not indicate which portion of the identification was correct) and let the game continue. At this stage, the turn of a team that has exhausted its pieces is taken as follows: the team member whose turn it is makes another attempt to identify all the strings.

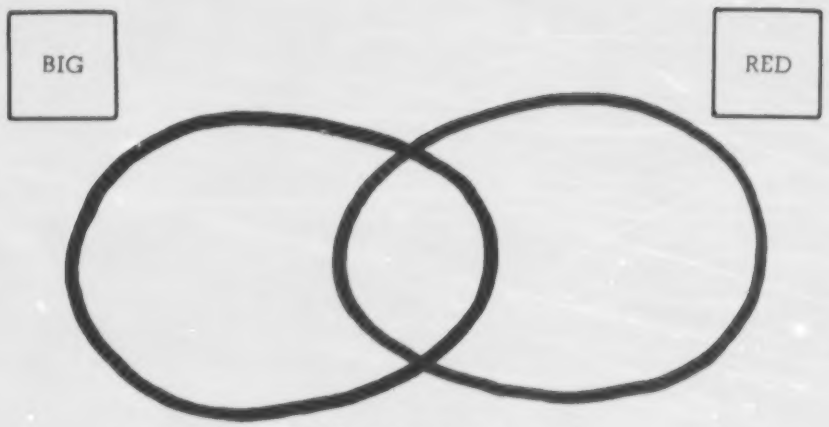
Version A: The String Game with A-Blocks

Use only the smooth (or only the rough) A-Blocks as game pieces. This means you will have 24 game pieces, 12 for each team. To begin with you should use the following nine string cards (as mentioned under "Equipment" above, you will need two sets of these):



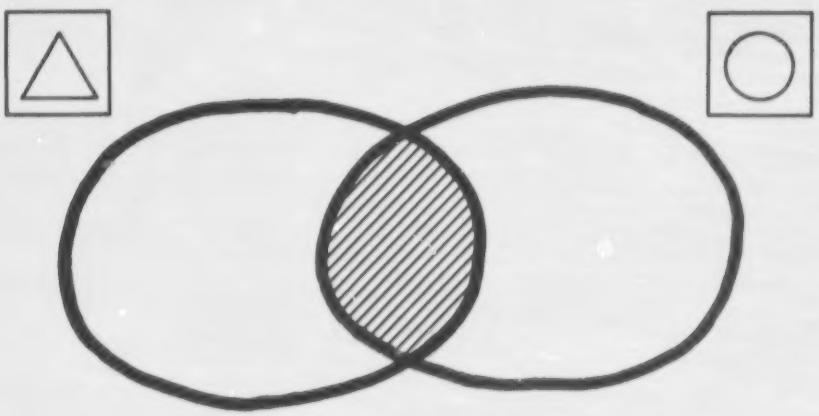
Variation #1 : Two Strings

Example 1 :



No Empty Regions

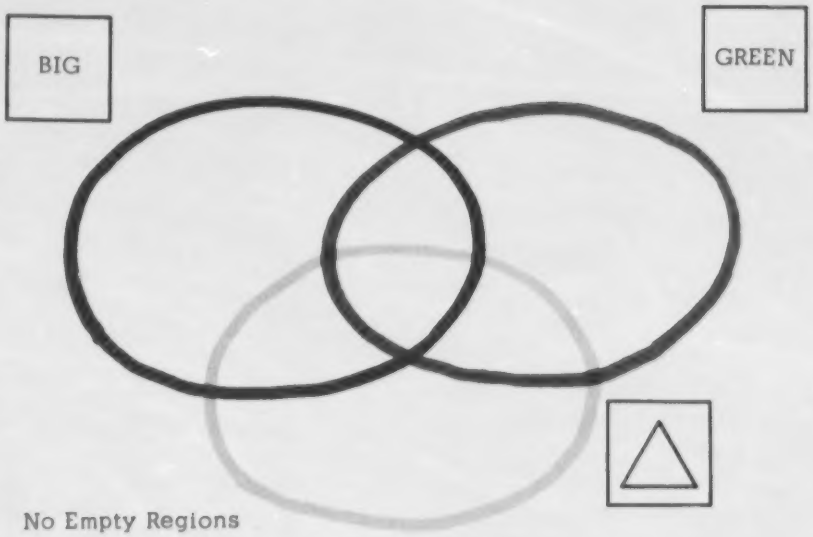
Example 2 :



One Empty Region

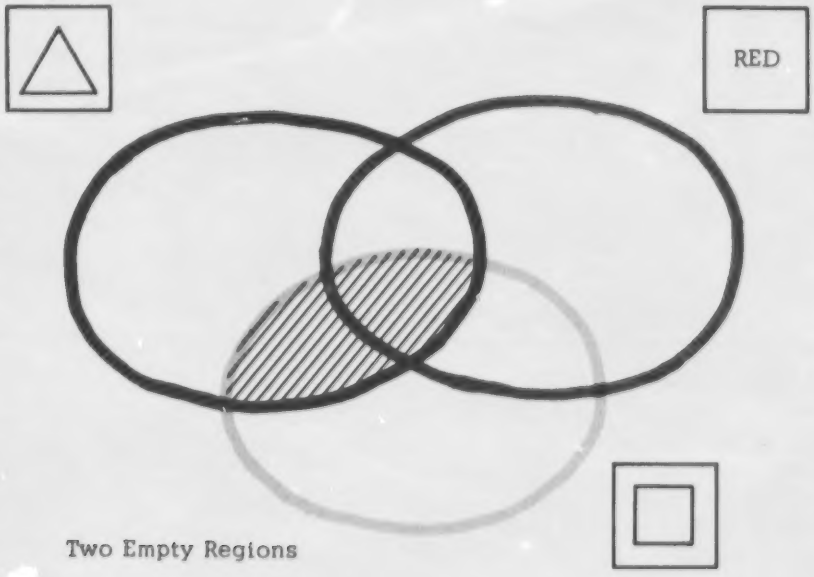
Variation #2 : Three Strings

Example 1 :



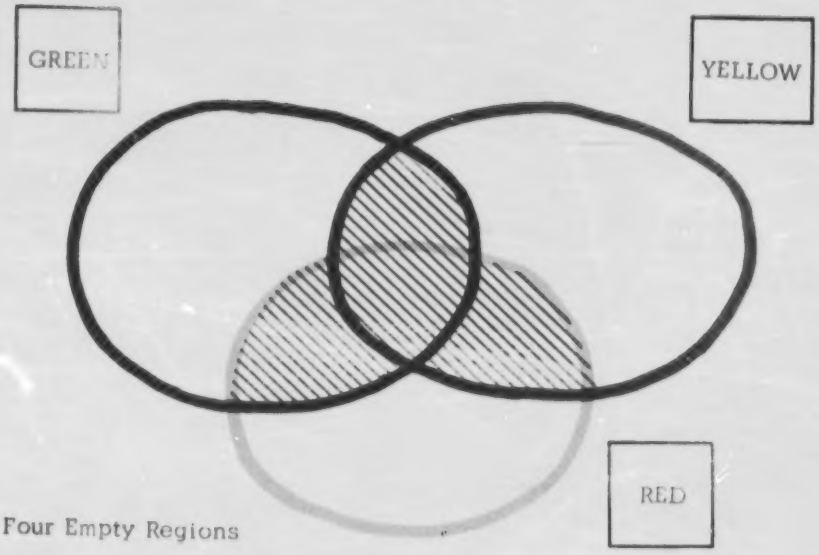
No Empty Regions

Example 2 :



Two Empty Regions

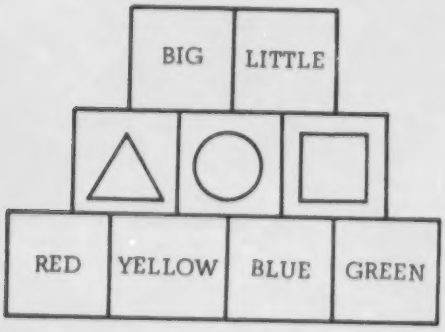
Example 3:

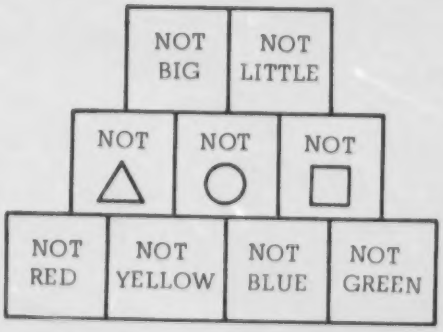


Four Empty Regions

Version B: The String Game with A-Blocks, using "Not" Cards

Once the children have become familiar with this simple version of the game, you can proceed to a more complicated version. It is recommended that the complicated version be obtained as follows: Increase your set of string cards by adding nine more cards, obtained from the original nine string cards by adding the word "NOT".

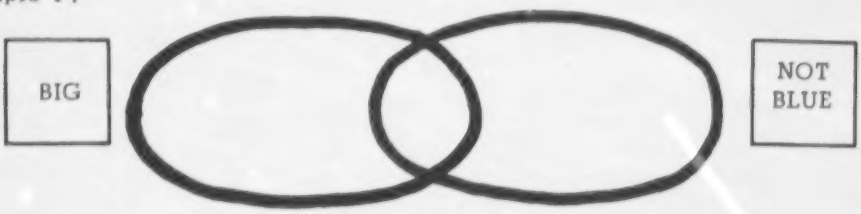




Add one set of these additional string cards to the nine already permanently displayed on the Game Board. Also make a face-down set to be added to the nine face-down cards you already have.

Note: Since "NOT BIG" and "LITTLE" describe exactly the same A-Blocks (and similarly for "NOT LITTLE" and "BIG"), whenever a child predicts that a given string card has one of these written on it when in fact it has the other, he or she should be counted right.

Example 1:

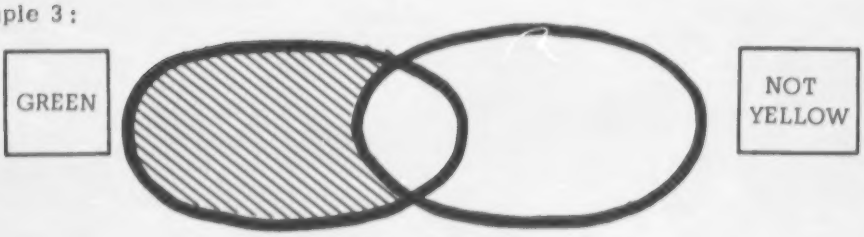


No Empty Regions

Example 2:

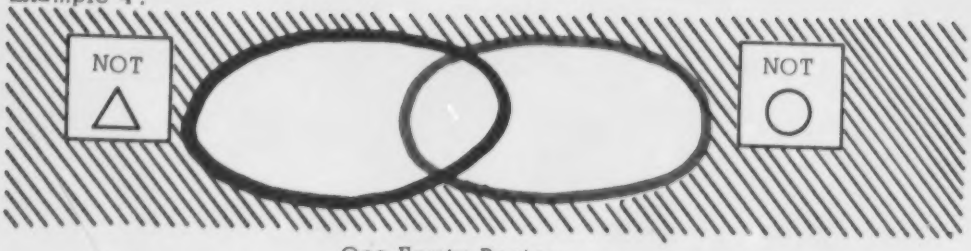
No Empty Regions

Example 3:



One Empty Region

Example 4:



One Empty Region

You should be warned that the "NOT" version of the game played with three strings is almost impossible to judge without a "crib-sheet" and is correspondingly difficult to play. Hence you would be well-advised to avoid it until such time as you think the two-string version is no longer challenging enough for the majority of your class.

Version C: The String Game with Numbers: Multiples and Order

Make 24 game pieces using the following numbers:

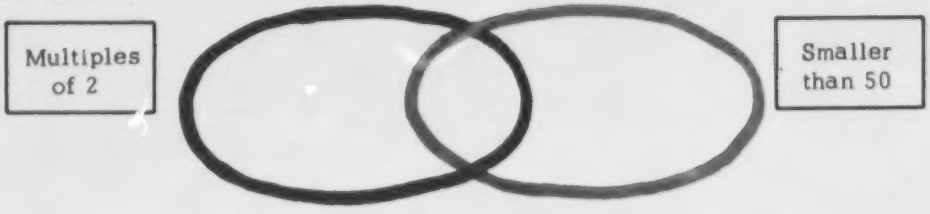
0	1	-5	-20	60	10
40	-55	27	-15	15	7
-12	24	100	-100	99	105
-80	-2	-1	30	45	-6

Use the following ten string cards (as mentioned under "Equipment" above, you will need two sets of these):

Multiples of 2	Multiples of 3	Multiples of 4	Multiples of 5	Multiples of 10
Odd Numbers	Larger than 50	Smaller than 50	Larger than -10	Smaller than -10

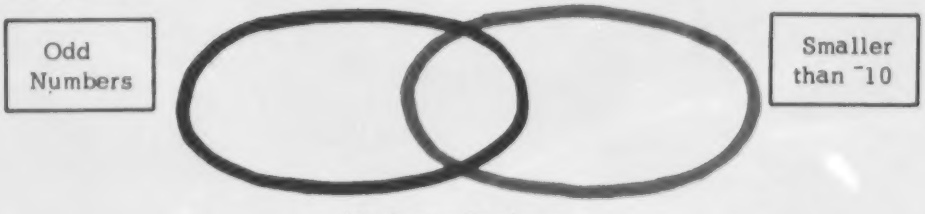
Variation #1 : Two Strings

Example 1 :



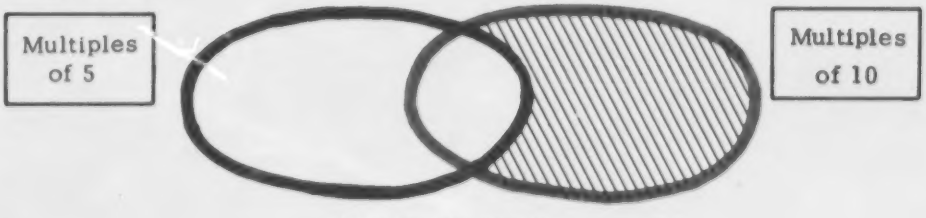
No Empty Regions

Example 2 :



No Empty Regions

Example 3 :

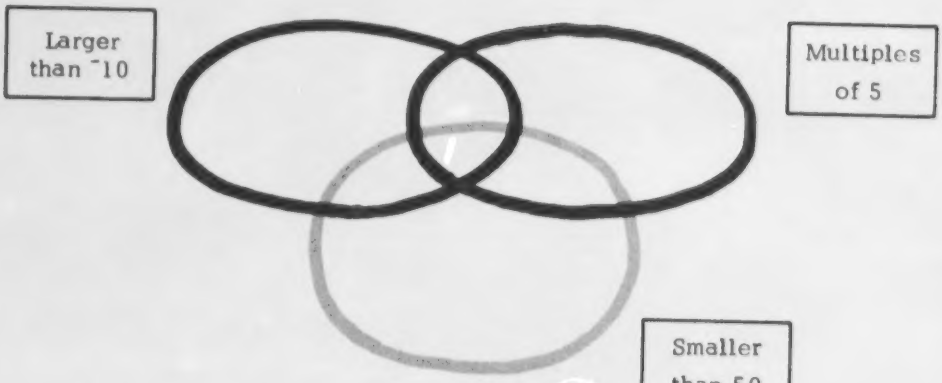


One Empty Region

The jump from the two-string version to the three-string version is much greater in the case of Version C of the String Game than it was in the case of Version A. Hence you should not expect to proceed to Variation #2 in this case as rapidly as you might have done with Version A.

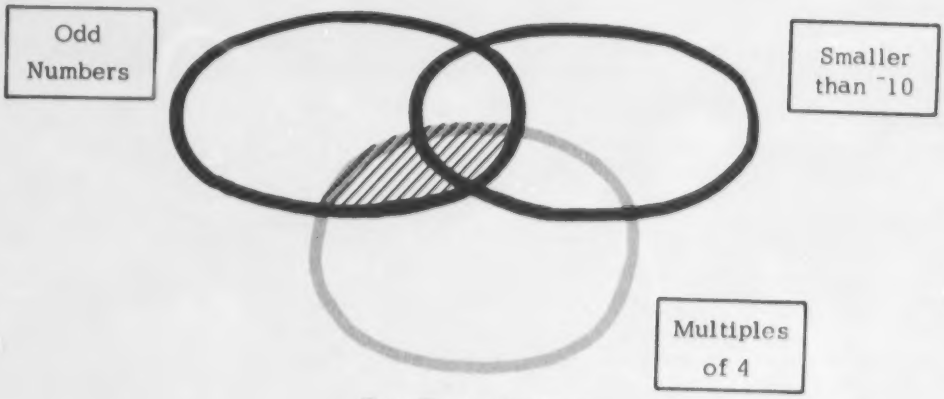
Variation #2: Three Strings

Example 1:



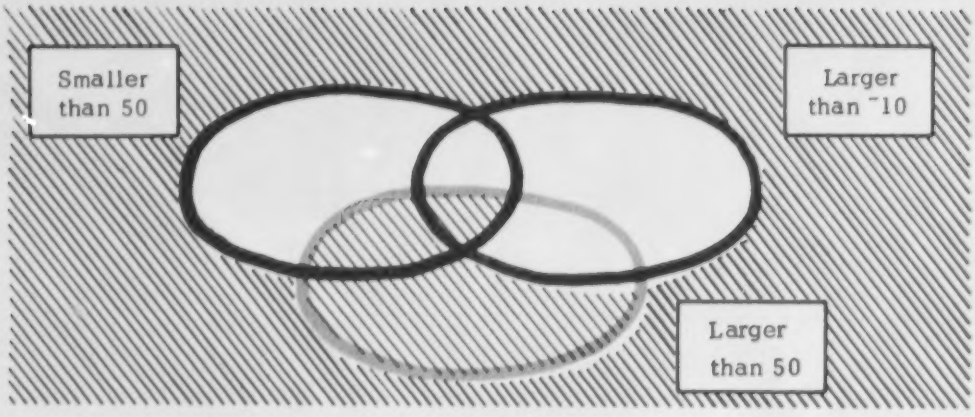
No Empty Regic

Example 2:



Two Empty Regions

Example 3:



Four Empty Regions

Of course, by hatching the "outside" region of this diagram we mean to indicate that there can never be any game pieces placed there correctly. Strictly speaking, that region is not empty, though. (The dot for the Eiffel Tower belongs there, for example.) However, our hatching is correct if we agree just to restrict ourselves to numbers.

Version D: The String Game with Numbers: Multiples, Divisors, and Order

Make 28 game pieces using the following numbers*:

9	45	105	40	⁻15	⁻5	99
7	10	⁻55	⁻1	27	18	5
20	0	100	6	2	1	3
4	24	8	12	⁻80	⁻100	60

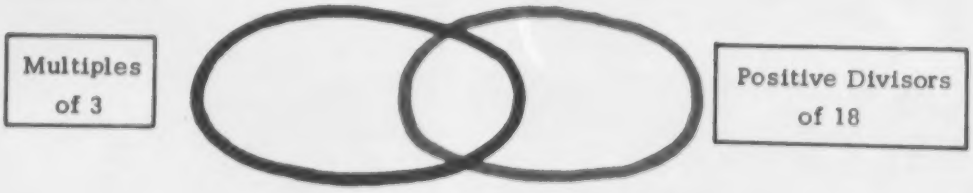
* Note that the game pieces and the string cards for Version D include those for Version C. Thus, the set for Version D is sufficient for both versions.

Use the following 16 string cards (as mentioned under "Equipment" above, you will need <u>two</u> sets of these):

Multiples of 2	Multiples of 3	Multiples of 4	Multiples of 5
Multiples of 10	Positive Divisors of 12	Positive Divisors of 18	Positive Divisors of 20
Positive Divisors of 24	Positive Divisors of 27	Larger than 50	Larger than -10
Smaller than 50	Smaller than -10	Odd Numbers	Positive Prime Numbers

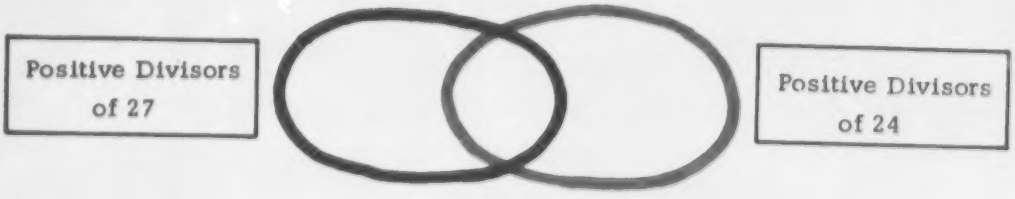
<u>Variation #1: Two Strings</u>

Example 1:



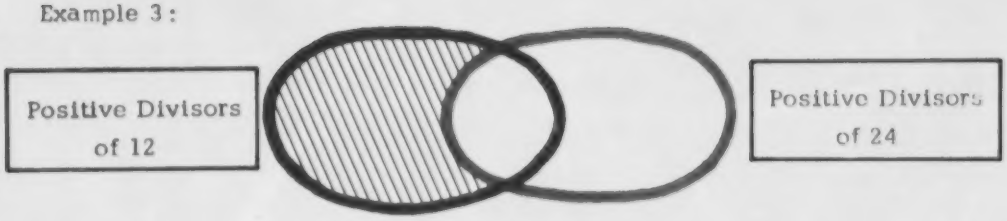
No Empty Regions

Example 2:



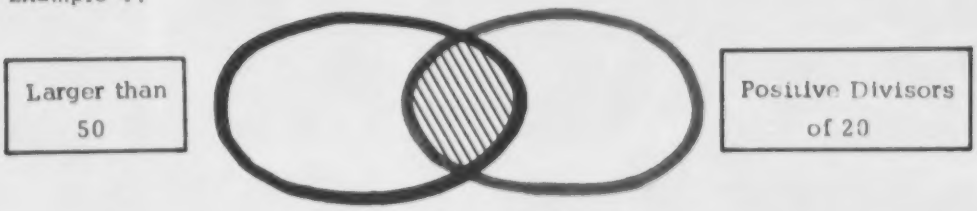
No Empty Regions

Example 3:



One Empty Region

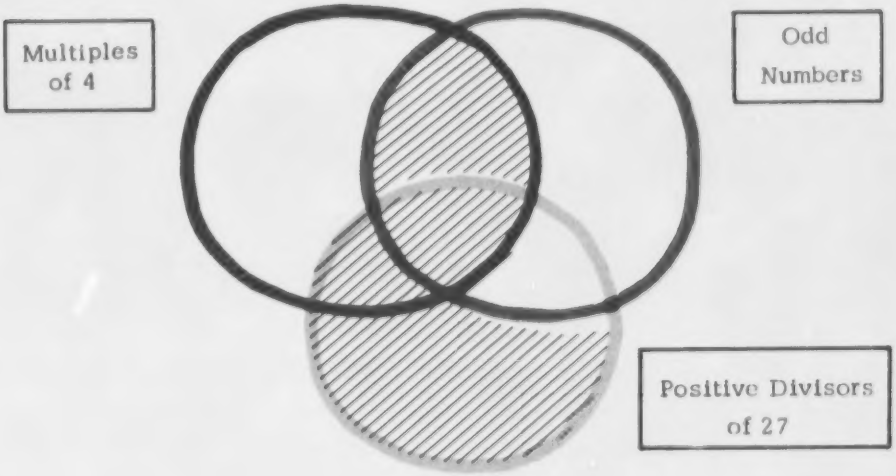
Example 4:



One Empty Region

The warning concerning the jump from the two-string version to the three-string version on page 45 applies even more so to Version D.

Example 5:



Four Empty Regions

2

LOGICAL THINKING

The language of strings is used as a vehicle to present various situations involving sets which stimulate logical thinking. The situations are, in many cases, similar to those experienced earlier in the CSMP program. In this chapter, however, explicit attention is paid to some of the precise terms in the English language used in discussing such situations. This is accomplished by asking the students to decide the truth or falsity of a number of statements concerning various sets. These statements use words such as: at least; at most; exactly; none of; each of; etc. In other situations students are asked to label dots in a string diagram so that each of a series of statements is true. No mastery is expected by the end of these lessons. The situations are rich, they encourage reasoning on the part of the students, and they draw upon the students' numerical and nonnumerical knowledge with the aid of string diagrams.

CAPSULE LESSON SUMMARIES

Section 2.1:	Using string pictures, consider the truth or falsity of statements involving "every", "no", "at least", "at most", "all", "each" and "exactly". (Worksheets L01, L02)
Section 2.2:	Given conditions involving the language emphasized in Section 2.1, label the dots in string pictures so that the conditions are met. (Worksheets L03, L04)
Section 2.3:	Extend the experiences of Section 2.1 to include situations involving more than one string.
Section 2.4:	Continue with exercises similar to those of Sections 2.2 and 2.3. (Worksheets L05, L06, L07, L08)

2.1 Logical Thinking #1

Exercise 1:

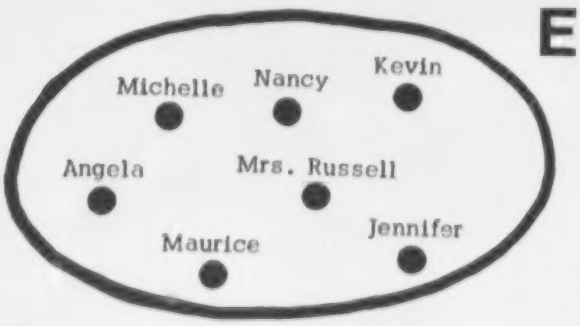
T: Someone tell me who is wearing glasses in this class today?

S: Michelle, Angela, Nancy, Maurice, Kevin, Jennifer, Mrs. Russell.

T: How many people is that?

S: Seven.

T: (Draw a red string on the board.) This red string is for these seven people who are wearing glasses today. We'll call this set "E". I'll also label the dots.



T: Now I will say some things about this situation and you must tell me whether what I say is true or false. Here's the first statement: Every member of E is a boy.

S: False. Michelle, Angela, Nancy, Jennifer, and you are girls.

T: No member of E is a boy.

S: False. Kevin and Maurice are boys.

T: At least one member of E is a boy.

S: True.

T: Explain your answer.

S: "At least one" means "one or more than one", and here we have two boys which is more than one.

T: There are at least two boys in E.

S: False. There are at least three boys in E.

T: What does "at least two" mean?

S: Two or more than two.

T: Is three more than two? (Yes.) So is "There are at least two boys in E" true or false?

S: True.

T: At most one member of E is a boy.

S: False.

T: First explain what "at most one" means and then explain your answer.

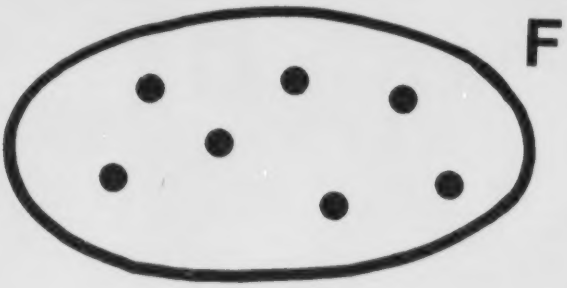
S: "At most one" means "one or less than one" (i.e., one or none). But there is more than one boy in E, so it's false.

Note: The phrase "at most" is very difficult for most students when first introduced. Gradually most of them master this language by experience and exposure to many different contexts in which it is used. Don't worry if some

students are still having problems with the idea of "at most" even after all the lessons on this topic for this semester have been taught. They will encounter this language regularly throughout the CSMP Intermediate Grades Program and eventually it will be a familiar part of their vocabulary. Have patience in this regard, but continually remind your students of the meaning of this phrase whenever it comes up in a natural way.

Exercise 2:

Draw on the board:



T: These dots are for whole numbers. I will now write a sentence on the board that tells you something about these numbers.

Write on the board:

> Every member of F is a whole number which is both larger than 20 and a multiple of 3.

T: What are some numbers that could be in F?

S: 15, because it's a multiple of 3.

S: But 15 is not larger than 20 so it doesn't belong.

S: 21.

T: Is it larger than 20? (Yes) Is it a multiple of 3? (Yes) How do you know?

S: $21 = 7 \times 3$.

S: 30, because 30 is larger than 20 and $30 = 10 \times 3$; so it's a multiple of 3.

S: 300.

T: How many times 3 is 300?

S: 100 (i.e., $300 = 100 \times 3$).

T: Is there a number larger than 300 that could be in F?

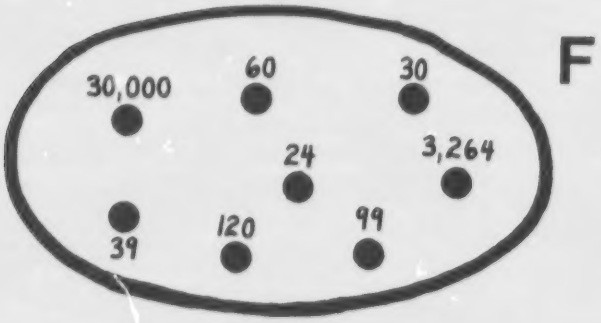
S: 303, 306, 309, ...

T: How many time 3 is 309?

S: 103 ($309 = 103 \times 3$).

T: Let's label the dots in the string diagram. Try to use numbers different from the ones we've already discussed if you can.

Have several students come to the board and label the dots. Be sure the class is in agreement for each label. Where there's disagreement ask the student who labeled the dot to explain his or her answer. We now assume that this is the picture you got. You will have to adjust the following dialogue according to your actual picture.



T: F is this set of exactly eight numbers. Nothing else is in F. I will now say some things about F and you have to answer "true" or "false" and explain your answers.

T: <u>All</u> numbers in F are even.

S: False. 39 and 99 are both odd.

T: <u>No</u> number in F is even.

S: False. All the rest (besides 39 and 99) are even.

T: <u>At least</u> three numbers in F are even.

S: True. "At least three" means "three or more than three" and there are six even numbers in F.

T: <u>At least</u> six numbers in F are even.

S: True. "At least six" means "six or more than six" and there are exactly six even numbers in F.

T: <u>At least</u> three numbers in F are odd.

S: False. There are only two.

T: At most one number in F is odd.

S: False. "At most one" means "one or zero" and there are two odd numbers in F.

T: At most four numbers in F are even.

S: False. "At most four" means "four or three or two or one or zero" and there are six even numbers in F.

T: At most four numbers in F are odd.

S: False.

T: What does "at most four" mean?

S: Four or less than four.

T: How many odd numbers are there in F ?

S: Two.

T: Is two less than four?

S: Yes. So it's true that there are at most four odd numbers in F.

There are two worksheets available for this lesson. Pass out L01 and do the first three or four problems collectively. Then have the students work on their own but caution them not to work too fast and to think carefully before answering. When they have finished L01, they should turn the sheet over and do L02.

As you move around the room observing the students' work, from time to time ask a given individual student to explain one of his or her answers to you. This may reinforce the attitude that they need to think about their answers and not just guess. If a particular question is answered incorrectly by many students, it may be useful to stop everyone for a few minutes and have a collective discussion on that problem.

After most of the students have had time to finish the two worksheets, go over the answers, discussing those statements for which there appears to be substantial disagreement. Below are explanations for the answers to the two worksheets; the actual answer keys are on page 60.

If there is still 10-15 minutes left for this lesson, play one round of the String Game, Version C. (One way to shorten the game is to put up a large number of pieces as original information at the outset rather than the usual one or two from each team's set.)

Explanations for the answers to L01 and L02

L01 :

1. 0, 8, ⁻16 are even.
2. 1, ⁻7, ⁻13, 5, 73 are odd.
3. Since three numbers in D are even, it is true that at least two are even. Remember "at least two" means "two or more than two".
4. "At least three" means "three or more than three".
5. 0 is neither positive nor negative.
6. There are three even numbers in D and "at most one" means "one or none".
7. "At most two" means "two, one, or none".

L02:

1. 10 and ¯11 are not multiples of 4.

2. 20, 100, ¯8 and 16 are multiples of 4.

3. There are <u>exactly</u> four multiples of 4, so surely there is <u>at least</u> one.

4. "At most one" means "one or none" and there are four.

5. See (2) above.

6. There are only two negative numbers and "at least three" means "three or more".

7. "At most three" means "three or two or one or none" and there are two.

8. 10 is even but <u>not</u> a multiple of 4.

Name _____

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L 01 *

D is a set of exactly eight numbers. Below the string are some statements about D. Circle T if the statement is true and F if it is false.

T (F) 1. No number in D is even.

T (F) 2. All numbers in D are even.

(T) F 3. At least two numbers in D are even.

(T) F 4. At least three numbers in D are even.

T (F) 5. Each number in D is either positive or negative.

T (F) 6. At most one number in D is even.

T (F) 7. At most two numbers in D are even.

Name _____

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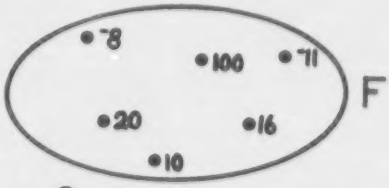
L 02 **

F is a set of exactly six numbers. Below the string are some statements about F. Circle T if the statement is true and F if it is false.



T (F) 1. Each member of F is a multiple of 4.

T (F) 2. No member of F is a multiple of 4.

(T) F 3. At least one member of F is a multiple of 4.

T (F) 4. At most one member of F is a multiple of 4.

(T) F 5. Exactly four members of F are multiples of 4.

T (F) 6. At least three members of F are negative.

(T) F 7. At most three members of F are negative.

T (F) 8. Every even number in F is a multiple of 4.

2.2 Logical Thinking #2

Exercise 1:

T: True or false? At least one of you watched TV last night?

S: True.

T: How do you know?

S: Because I watched my favorite show, Happy Days.

T: How many of you watched TV last night? Five of you. At most two of you watched TV last night.

S: False. "At most two" means "two or one or none" and there were five.

T: No one is absent today.

S: False. Beth, Lauren, and Mike aren't here.

S: Also Dirk and Edward are absent.

T: That's right. There are five absent and 22 present. At most eight students in this class are absent today.

S: True. "At most eight" means "eight or less" and five is less than eight.

T: At least eight students are absent today.

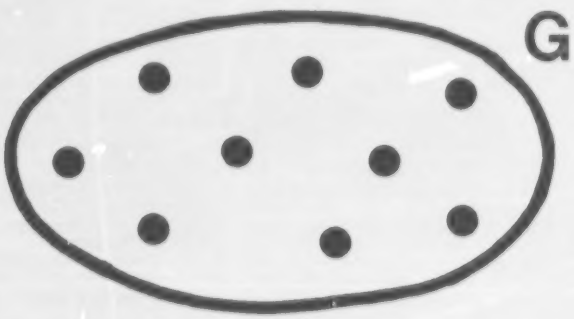
S: False. Five is not eight or more.

T: Exactly eight students are absent today.

S: False. there are exactly five absent.

Exercise 2:

Draw the following on the board:



T: G contains exactly nine numbers. I'm going to tell you something about these nine numbers and then we'll try to label the dots to agree with this information.

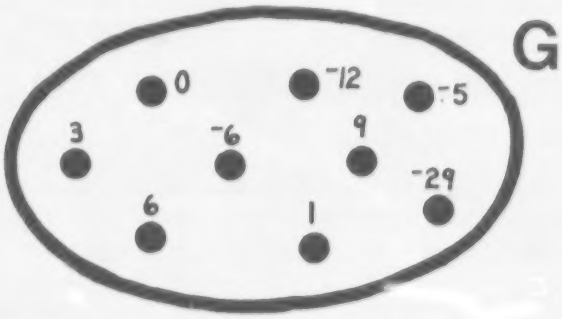
Write on the board:

> There are at least four multiples of 3 in G
>
> and
>
> There are at most two even numbers in G
>
> and
>
> There are exactly five negative numbers in G
>
> and
>
> There are no numbers in G larger than 10.

T: Copy the picture on your piece of paper and try and label the dots so everything written on the board is true.

When a student thinks he or she has solved the problem, have him or her put his or her solution on the board and discuss it collectively. If it's not exactly correct, try to get a correct solution from it by changing as few labels as possible. The following dialoque is an example of how this discussion might go.

T: Sally, put your solution on the board.



T: What do you think of Sally's solution?

S: It's not correct. She has four even numbers and there are to be at most two even numbers in G.

T: What are the four even numbers?

S: 6, $^{-}6$, $^{-}12$, 0.

S: Also G is supposed to have exactly five negative numbers and there are only four here.

T: Does G have at least four multiples of 3?

S: Yes. There are six multiples of 3 (0, 3, 6, $^{-}6$, $^{-}12$, 9) and six is at least four.

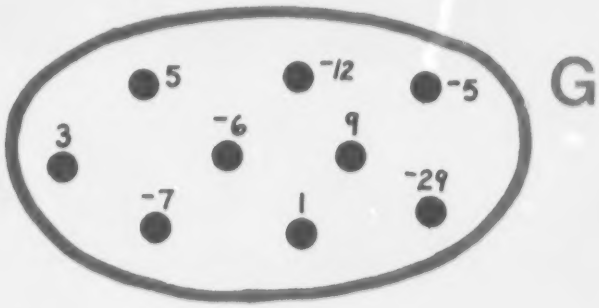
T: Are there any numbers larger than 10 in Sally's solution?

S: No.

T: Can someone fix Sally's solution so it is correct but by only changing the labels of a few dots?

S: I can do it by changing only two dots—change "6" to "⁻7" and "0" to "5".

Fix the picture at the board:



T: Who can explain why it is correct now?

S: Now it's OK. There are exactly four multiples of 3 (3, ⁻6, 9, ⁻12) and so there are at least four. There are exactly two even numbers (⁻12, ⁻6) and so there are at most two. There are exactly five negative numbers (⁻7, ⁻6, ⁻12, ⁻5, ⁻29) and 9 is the largest number in G so none of them are larger than 10.

Pass out worksheets L03 and L04. Remind the class that most likely each student will have a different solution. Tell them to raise their hands when they think they have a solution to L03 and you will try to move around and check them. While they're waiting for you they should proceed to L04. After most students have finished L03, have one or two put their solutions on the board. You might follow a similar procedure for L04, if there's time and the interest still seems high.

- Have the class determine if the solutions on the board are correct and discuss any objections that are raised. If you wish, you could make up a few true-false questions involving "at least", "at most", "exactly", etc., for some of the solutions on the board to be answered collectively.

- Try to leave a few minutes at the end of the lesson to play a round of the String Game, Version C. (See page 58 for a suggestion to shorten the game if there's not much time.)

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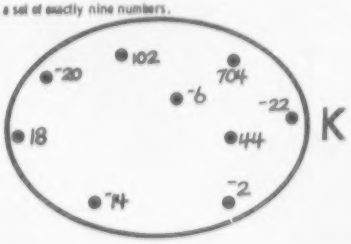
Name _____ L 03 *

K is a set of exactly nine numbers.



Label the dots so that:

each number in K is even

and

at least two numbers in K are negative

and

at most one number in K is a multiple of 5.

L03

(Many other solutions are possible.)

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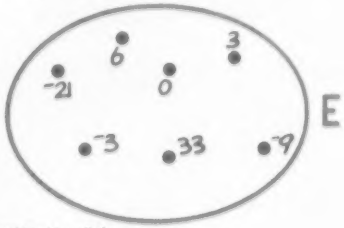
Name _____ L 04 **

E is a set of exactly seven numbers.



Label the dots so that:

each number in E is a multiple of 3

and

at most two numbers in E are larger than 7

and

at least four numbers in E are odd.

L04

(Many other solutions are possible.)

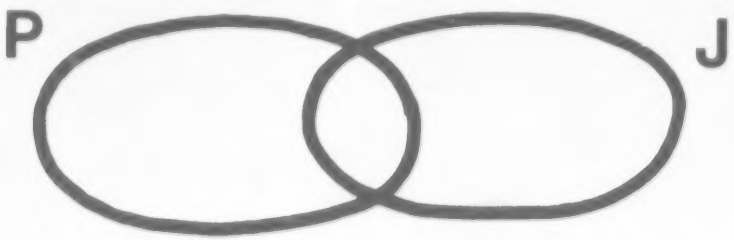
2.3 Logical Thinking #3

Exercise 1:

Draw the following on the board:

T: A fourth grader named "Jim" lives next door to me and he has lots of friends. This green string (J) is for all of Jim's friends. Some of Jim's friends have a pet. Who would like to draw a red string (P) for all of Jim's friends who have pets?

Have a student come to the board and draw such a red string.

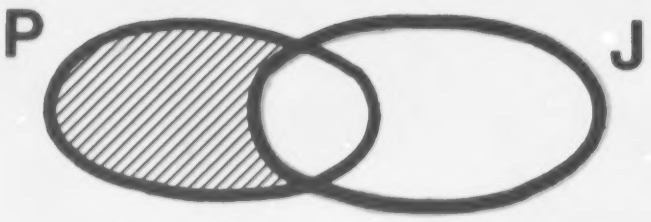


T: What do you think of this string picture?

S: OK.

S: We can hatch some of it.

T: Come to the board and do it.

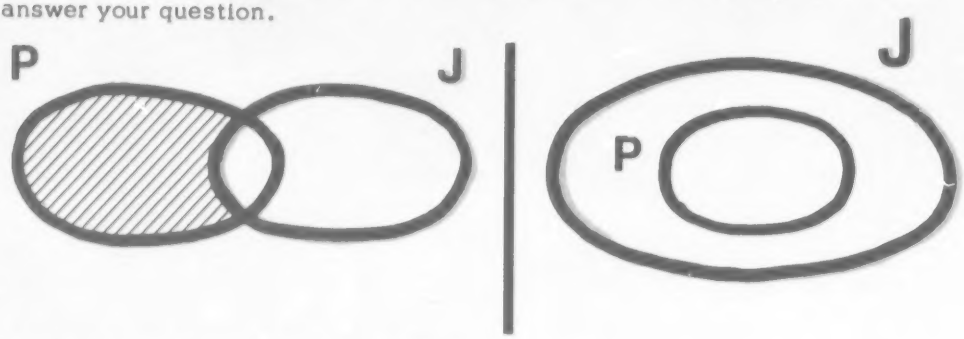


T: Explain why you hatched that part.

S: Because somebody in there would not be one of Jim's friends but P is for Jim's friends who have pets.

T: Very good thinking. You are completely correct. Is there another way to draw the red string so we get the same information but this time without hatching?

Leave the picture on the board and draw another green string on a different board if possible. Then call on someone to draw in the red string so as to answer your question.



T: What do you think of this drawing?

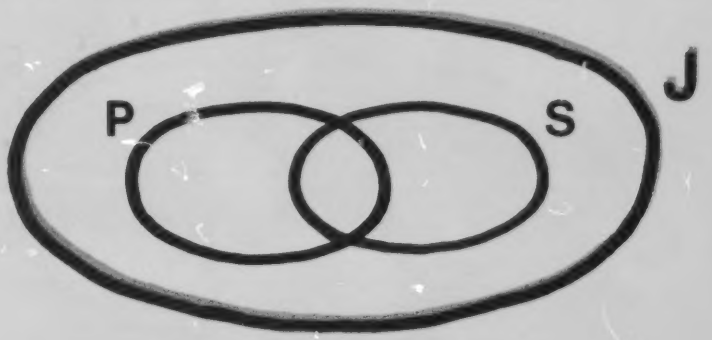
S: It gives the same information as the hatching because if a dot is in the red string (P) it is also in the green one (J).

T: Excellent. So we have a choice of the two pictures. I'm going to keep the second one for the rest of this exercise.

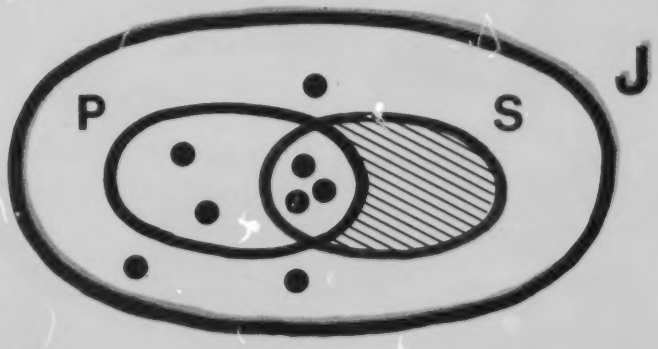
Erase the first one.

T: I'd like a blue string (S) for all of Jim's friends who have sisters.

Have a student come to the board and draw such a blue string. Be sure the red and blue strings overlap. If they don't, ask where to put a dot for a friend of Jim's who has a pet and a sister. Also be sure the blue string is completely inside the green one. If not, remind them that every member of S is a friend of Jim's and so we can draw the blue string inside the green one for the same reason we could draw the red one inside the green one.



T: Very good. Now I will add some information to the picture. As I do so, see if you can tell what new information I've added.



T: I'll tell you that all the dots are drawn. Jim has exactly eight friends. What else can you now tell about Jim's friends?

S: Two friends of Jim's have pets but not sisters.

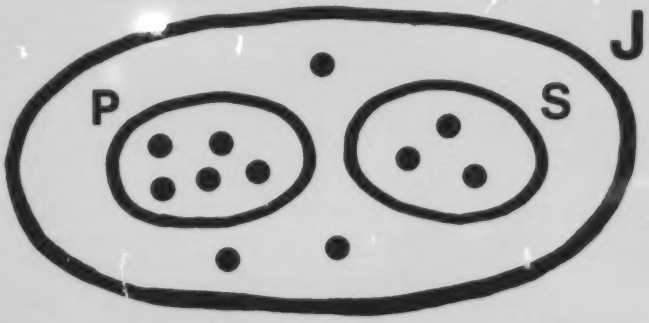
S: Every friend of Jim's who has a sister also has a pet.

S: Three friends of Jim's don't have a pet or a sister.

T: Is there another way to draw the picture without hatching but so that it tells the same information?

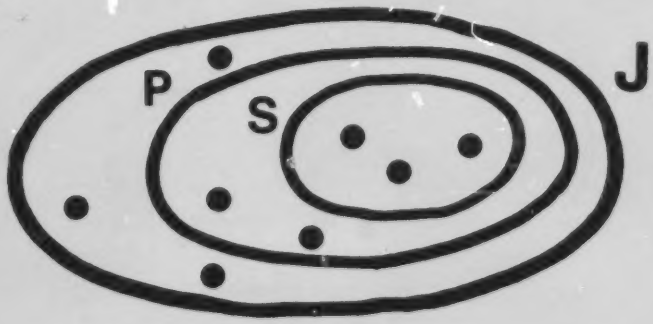
Have a student draw such a picture on the board.

First attempt:



S: That's not right. Now Jim has 11 friends but he really has only eight.

Second attempt:



S: This is correct because everybody in S is in P.

Leave both pictures on the board for the rest of this exercise. For some of the following questions it may be useful to have the student who answers come to the board and point to the appropriate dots during his or her explanation.

T: True or false? Jim has <u>at least</u> one friend who has a pet but not a sister.

S: True. There are two such friends and "at least one" means "one or more than one".

T: <u>At most</u> one of Jim's friends has a sister.

S: False. All three friends in the middle have sisters.

T: <u>Every</u> friend of Jim's who has a pet has a sister.

S: False. There are two friends with pets but not sisters.

T: <u>Every</u> friend of Jim's who has a sister has a pet.

S: True. Because of the shading.

T: Every friend of Jim's who does <u>not</u> have a sister has a pet.

S: False. The three outside both the red and blue strings do not have pets and also don't have a sister.

T: <u>At least</u> five of Jim's friends don't have sisters.

S: True. Exactly five don't have sisters.

In the time remaining for this lesson play the String Game, Version C.

2.4 Logical Thinking #4

Exercise 1:

Pass out worksheet L05.

T: Read at the top of the page what the three strings are for.

Ask a student to read it aloud for the benefit of those who may have difficulty reading.

T: Do you think there are only four students in Max's school?

S: I guess so, since there are only four dots but that would be a very small school.

T: I agree, but really there are many more than four students—too many for me to draw all the dots. So I've only drawn dots for four of them. Is there anything we can say about the number of students in Max's school?

S: There are at least four.

S: There are at least three, at least two, at least one.

T: Can we say "there are at most four"?

S: No, because you said you didn't draw all the dots and there are already four dots there.

T: What's the answer to the first question?

S: True, because of the shaded part.

T: The answer to the second question?

S: False. There's a boy (in the middle) who wears glasses.

Now let the students finish the worksheet on their own. After a few minutes, discuss the answers.

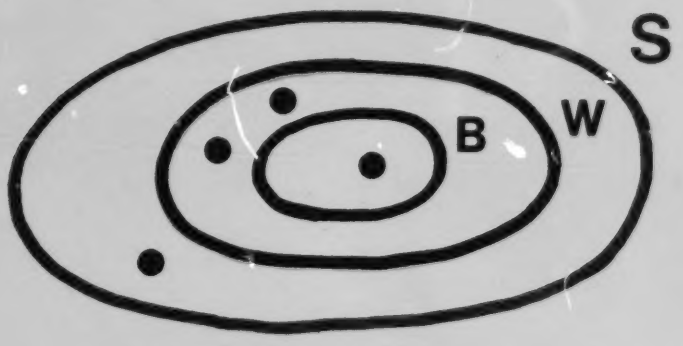
3. True, in fact there are <u>at least</u> two such girls since there are two dots in W but not in B.

4. False. In (3) above we argued there were at least two such girls.

5. True. The dot outside both B and W must be for a girl who does not wear glasses.

T: Can someone draw another picture of the same situation which does not use hatching?

Have someone do so at the board:



Discuss this solution if you think it would be useful to do so.

Have the students turn the worksheet over to L06 and work on it individually. L07 and L08 are available for those who finish quickly. After most students

have had time to finish L06, go over the answers and discuss those problems for which there is substantial disagreement. Have the students explain a few of the answers as you go over them. If there's time have some students put their solutions to L07 and L08 on the board and discuss them with the rest of the class.

Play the String Game, Version C for the rest of the lesson, if time permits.

Explanations for the answers to L06 are given below. The actual answer key is on page 75.

L06:

1. False. The dot in the green string is for a Ford on this lot which is not red.
2. True because of the hatching.
3. From the previous statement we know there are no red Fords on this lot. Hence the statement is false.
4. "At most one" means "one or none" and since there are no red Fords, this statement is true.
5. False. The three dots outside the green string are for non-Fords.
6. True. There are at least three such cars since there are three dots outside the blue string. Hence there is at least one such car.
7. False. The three cars outside the blue string are not red and "at most one" means "one or none".
8. True. There are at least two such cars since there are two dots outside both the blue and green strings. Hence there is at least one such car.

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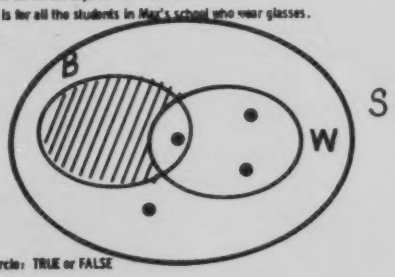
Name _____ L 05 *

S is for all the students in Max's school.
B is for all the boys in Max's school.
W is for all the students in Max's school who wear glasses.



Circle: TRUE or FALSE

(T)	F	1. Each boy in Max's school wears glasses.
T	(F)	2. Each student in Max's school who wears glasses is a girl.
(T)	F	3. At least one girl in Max's school wears glasses.
T	(F)	4. At most one girl in Max's school wears glasses.
(T)	F	5. At least one girl in Max's school does not wear glasses.

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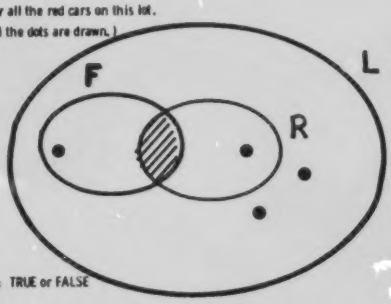
Name _____ L 06 **

L is for all the cars on our school's parking lot.
F is for all the Fords on this lot.
R is for all the red cars on this lot.
(Not all the dots are drawn.)



Circle: TRUE or FALSE

T	(F)	1. On this lot, each Ford is red.
(T)	F	2. On this lot, no Ford is red.
T	(F)	3. On this lot, at least one Ford is red.
(T)	F	4. On this lot, at most one Ford is red.
T	(F)	5. On this lot, at most one car is not a Ford.
(T)	F	6. On this lot, at least one car is not red.
T	(F)	7. On this lot, at most two cars are not red.
(T)	F	8. On this lot, at least one car is both not a Ford and not red.

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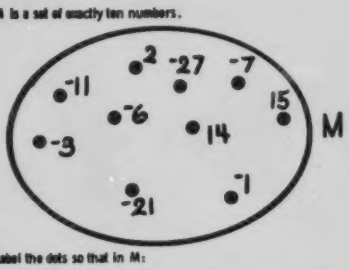
Name _____ L 07 **

M is a set of exactly ten numbers.



Label the dots so that in M:

each number is between -30 and 20
and
no number is a multiple of 4
and
exactly three numbers are even
and
at most four numbers are positive.

L07

(Many other solutions are possible.)

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Name _____ L 08 ***

C contains exactly seven numbers.

Label the dots so that:

at least four numbers in C are multiples of 2
and
at least five numbers in C are multiples of 3
and
at most two numbers in C are even
and
every number in C is a whole number smaller than 22.

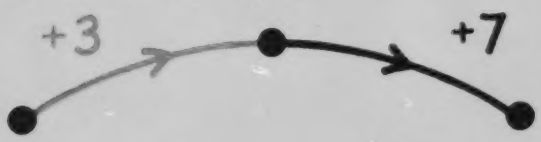
Footnote: With the given information there is only one solution, of course.

3

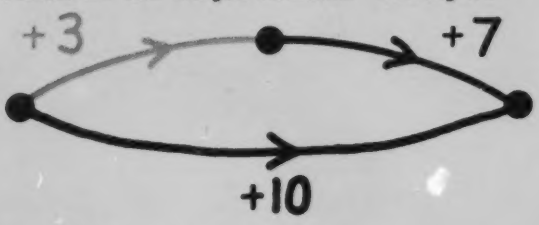
COMPOSITION GAMES

The concept of composition of relations is a notion that is deeply fundamental not only in mathematics but in many aspects of life. It is a notion that students have experienced over and over again in their CSMP mathematics program; also, in their experiences outside the classroom the concept is implicitly a part of much of their language and thinking.

In CSMP mathematics they have become accustomed to the idea that, for example, "adding 3 and then adding 7"



gives the same result as the single relation "adding 10"



regardless of what number is at the starting dot of the yellow arrow. In such an arrow picture, the student will draw a red arrow starting at each dot from which a yellow arrow starts and ending at the dot at which a blue arrow ends, provided they see a dot where the yellow and blue arrows are connected, as in the figure above.

Outside the classroom many common words and ways of thinking involve this same concept. For example, the relation, "you are my uncle" is the

composite of the relations "you are my parent" and "you are my brother".

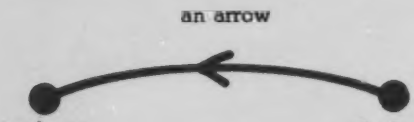
That is, my uncle is a brother of my parent.

In this chapter we concentrate explicitly on the idea of the composite of relations. Instead of dealing with specific relations, such as "+3" and "+7", or "you are my parent" and "you are my brother", we will let the student explore the idea itself, where the relations are not specified but are identified only by the colors of the arrows. In this way the experiences they have had in previous parts of the program will be brought together and focused on the idea itself.

In the lessons in this chapter, the colors are essential and must be easily distinguishable from each other by every student in the classroom. Be sure the colored chalk you use to make the suggested drawings shows up clearly on the chalkboard. Also, take care to draw particularly large dots and heavy arrows. If you have a black chalkboard, the blue chalk should be a light blue. Remember that the drawings suggested in these lessons are printed in the guide with black dots. Of course, you will reproduce these dots on your dark chalkboard with white chalk. Also, the arrows or loops that are printed in black in this guide should be drawn on the chalkboard with white chalk.

You will find it convenient to adopt language that is consistent throughout these lessons. Any change from one word to another in describing the dots

and arrows will be confusing. We suggest this language:



<u>end</u>, or the dot at which the arrow <u>ends</u>.

<u>start</u>, or the dot at which the arrow <u>starts</u>.

an arrow forming a loop.

start of arrow <u>and</u> end of arrow are the same dot so we don't need an arrow head.

CAPSULE LESSON SUMMARIES

Section 3.1: Given a rule for the composition of two relations, draw the composite arrows according to the rule.
(Worksheets L09, L10, L11)

Section 3.2: Extend the experiences of Section 3.1 to include two composition rules for the same arrow picture.
(Worksheets L12, L13, L14)

Section 3.3: Consider the special case in which a relation is composed with itself and add rules for the composition of a relation with itself three and four times.

Section 3.4: Extend the examples in Section 3.3 to consider cases involving the composition of a relation with itself more than four times.

3.1 Composition Games #1

If it is possible to draw the arrow picture on page 83 on your chalkboard before the class starts, the lesson can proceed immediately. If not, let the children watch you draw the arrow picture. Accept any comments about it that they might make, replying only that we shall be playing a game with the picture.

T: We shall play a game with these yellow and blue arrows. The object of the game is to draw red arrows, after you learn the rules of the game.

Move to the upper left portion of the picture you drew:

(The above picture is the upper left portion of the larger arrow picture on page 83.)

As you explain the rules of the game (as stated in the left-hand column) make the motions described in the right-hand column.

T:	This is the rule of the game...	Place your left forefinger on a dot at which a yellow arrow starts.
	Each time there is a dot from which a yellow arrow starts...	With your left forefinger in place at the dot, follow the yellow arrow with your right forefinger in the direction of the arrowhead.

T:	... and the yellow arrow ends at a dot from which a <u>blue</u> arrow starts...	Stop the motion of your right forefinger at the middle dot, tap the dot, and then follow the blue arrow.

T: ... then you must draw a red arrow from the dot where the yellow arrow starts...

Tap this dot several times with your left forefinger.

... to the dot where the blue arrow ends.

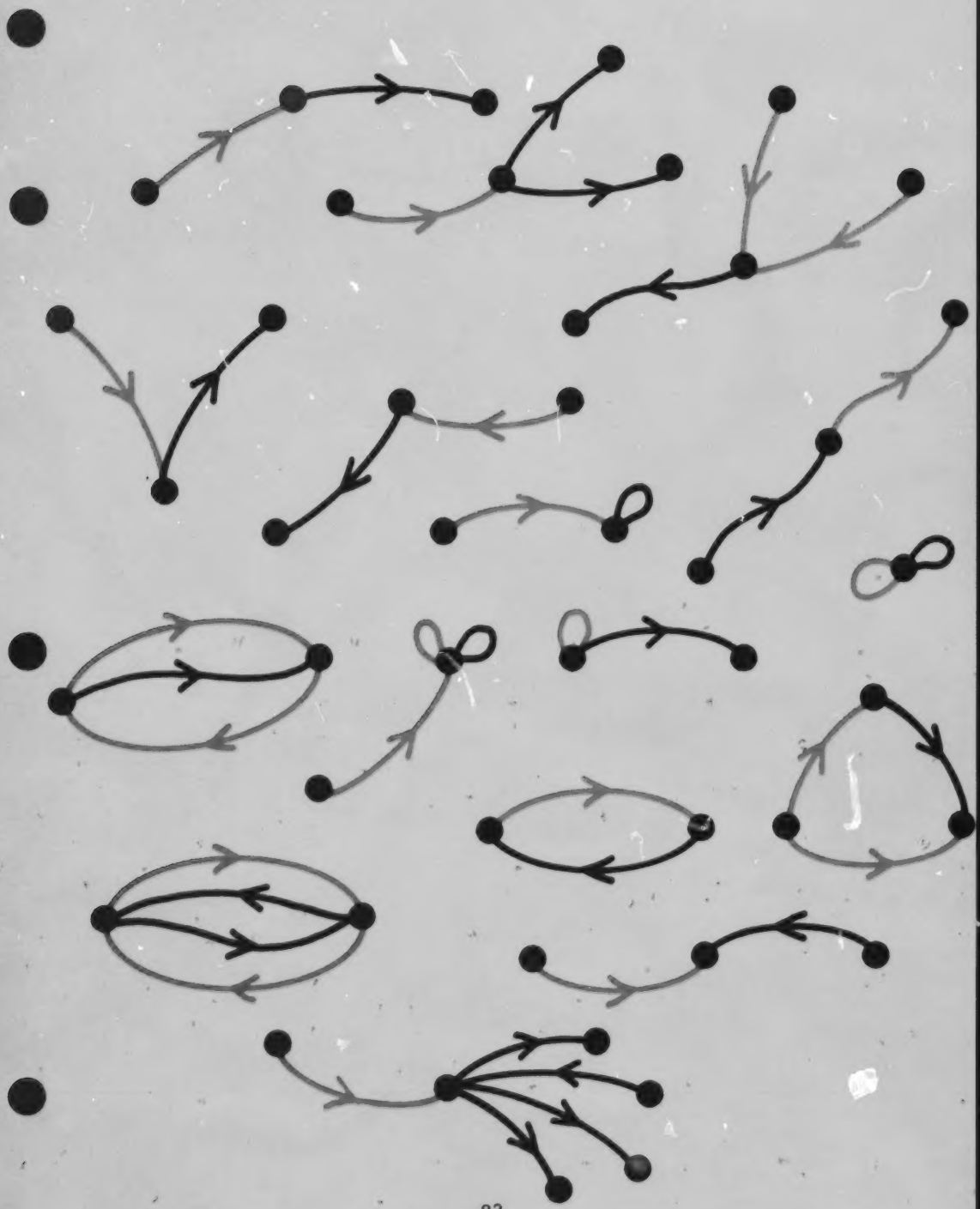
Tap this dot several times with your right forefinger.

Then draw the required red arrow.

T: Each time you draw such a red arrow, first trace it with your finger like this...
Don't draw a red arrow until you are sure it belongs there. Otherwise you will spoil the design of the picture.

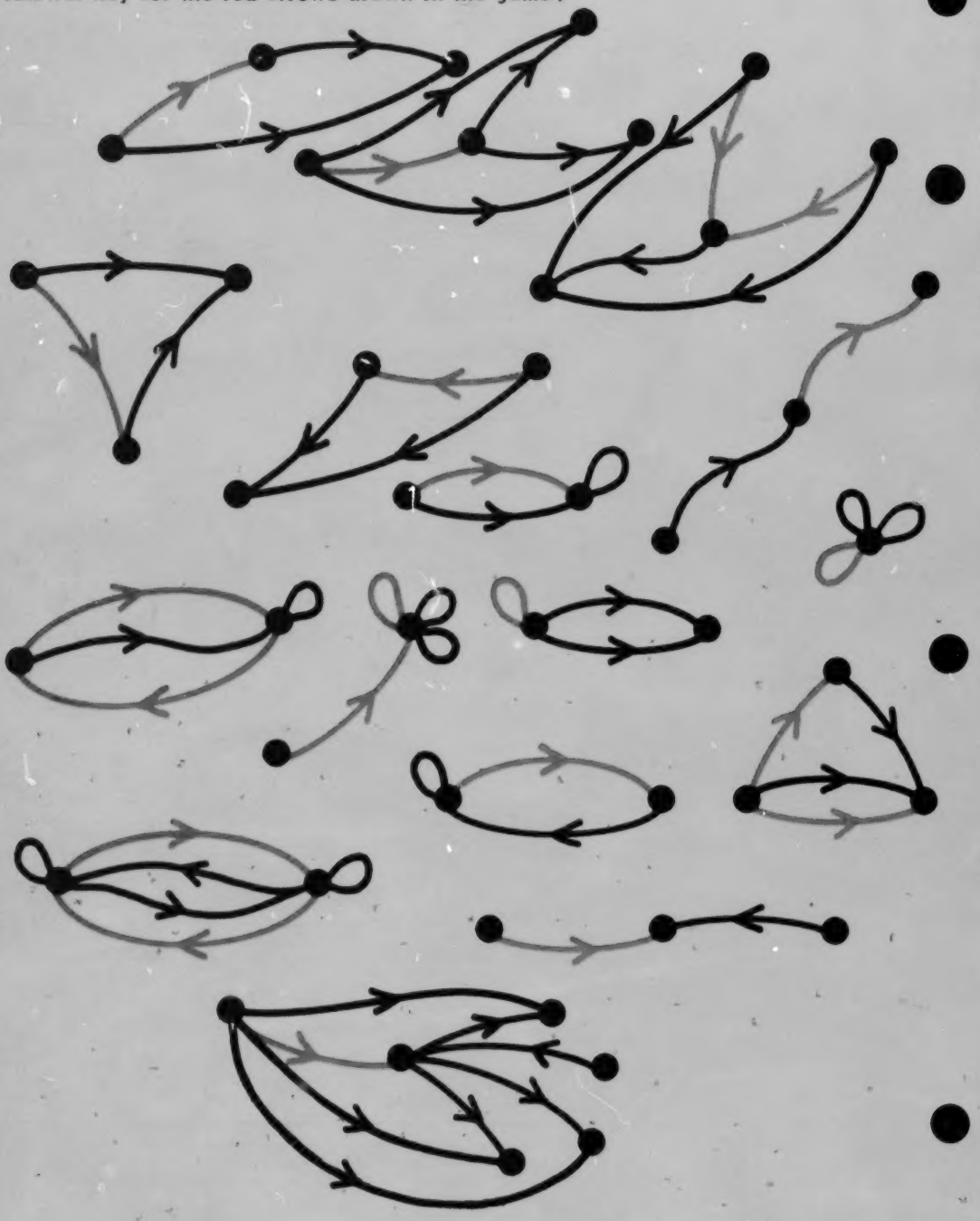
Repeat the statement of the rule, using the same words and gestures as before.

Now let the students take turns coming to the chalkboard and telling where a red arrow must be drawn, tracing it first with their finger, and then drawing it in red.



83

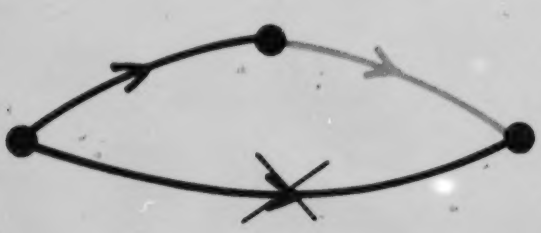
Answer key for the red arrows drawn in the game :



The drawing on page 83 was designed carefully in order to place some of the easier situations on the left. You have already drawn one of the red arrows on the left and the first students to come to the board will probably start on that side too. (On page 84 all the red arrows are given.)

If you think it desirable, you can make a competitive game out of it, with each side alternately sending a player to the board to find and draw a red arrow. Each would then get only one chance. If a player suggests a red arrow where it does not belong, say only that he or she did not follow the rule. If a player draws a red arrow where it does belong, his or her team gets a point.

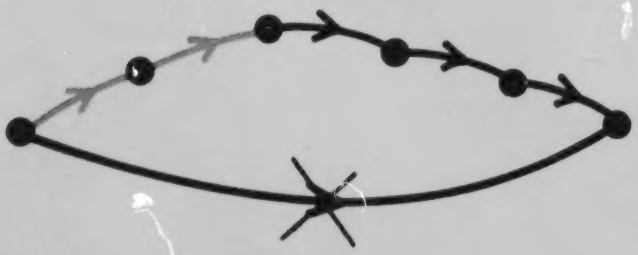
Note on common mistakes and difficulties: The most common mistake at first will occur when students confuse the rule "yellow...then blue" with "blue...then yellow":



Another mistake might occur when the directions of the arrows are not correctly followed:

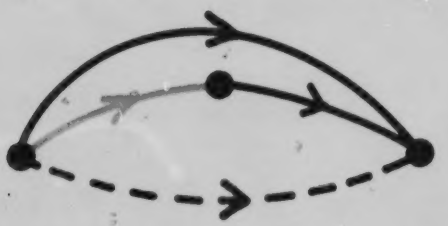
You might remind students making such mistakes that if they were driving on one-way streets and didn't follow the arrows, they would be whistled at by the policeman. Later on, if a similar mistake is about to be made, some pupils might say, "Whistle".

Still another common mistake (particularly on Worksheet L09, at the end of the lesson) is to mistake the rule "yellow...then blue" for "some yellows... then some blues".



The rule is "ONE yellow...then ONE blue".

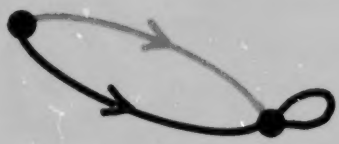
If it happens that a student looks at a portion of the arrow picture and proposes to draw a red arrow as indicated by the dotted arrow,



another student will probably remark that the red arrow has already been drawn. There is nothing wrong with the dotted arrow; it is merely redundant. That portion of the picture is complete without it.

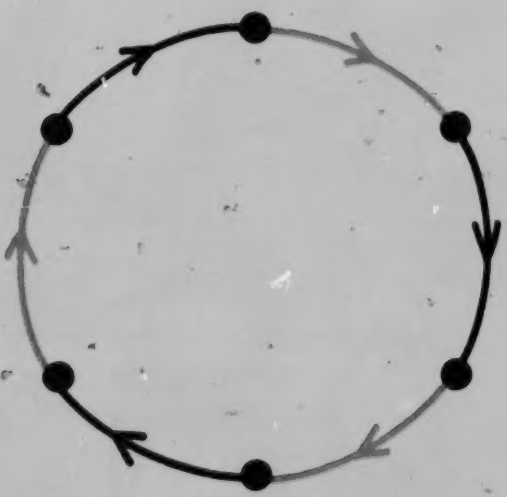
On the center and right side of the arrow picture, some students will have difficulty with the loops. The first time this occurs you might ask the

student to use his two forefingers, indicating with his left hand the dot at which the yellow arrow (or loop) starts and using his right hand to trace the yellow arrow (or loop) and then the blue arrow (or loop). The left forefinger will then be at a dot where a red arrow starts and the right forefinger will be at the dot where that same arrow ends.



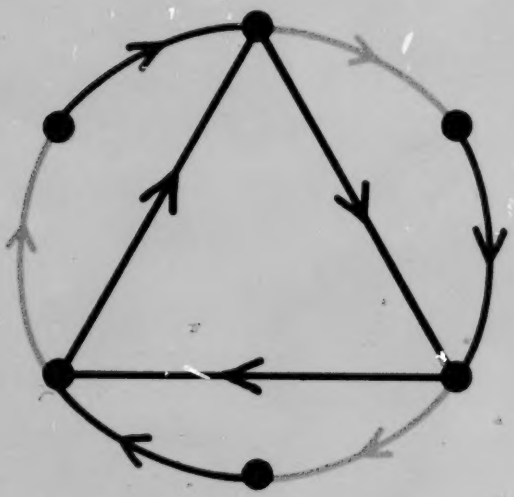
It is a good idea to ask students to say "yellow...then blue" each time as they trace the two arrows (or loops).

After all the red arrows are drawn on the arrow picture, or after the game is over, draw this arrow picture on the board:



T: What will the red arrows form in this picture?

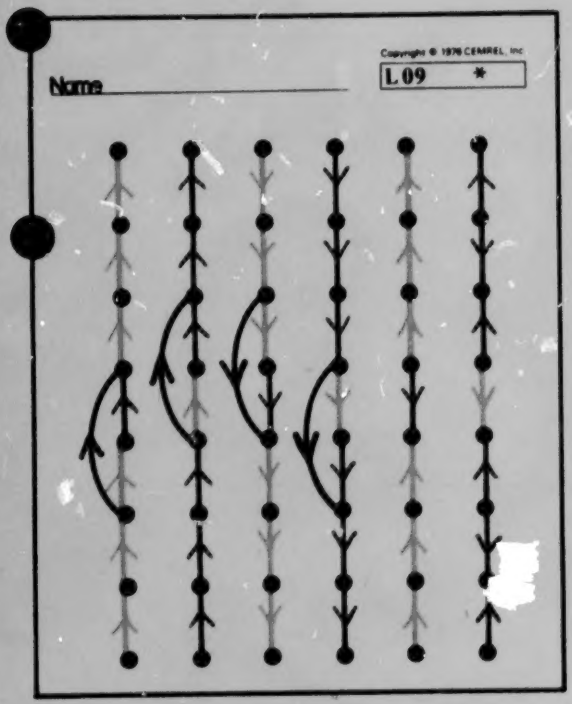
Accept any reasonable answers. Some students may answer: a triangle. Then ask some students to come to the board to draw the red arrows. The result might look like the triangle in the following picture, if the red arrows are drawn "inside" the "circle". But the red arrows could be drawn outside the circle, or some inside and some outside. Accept any of these, provided the red arrows start and end at the proper dots.

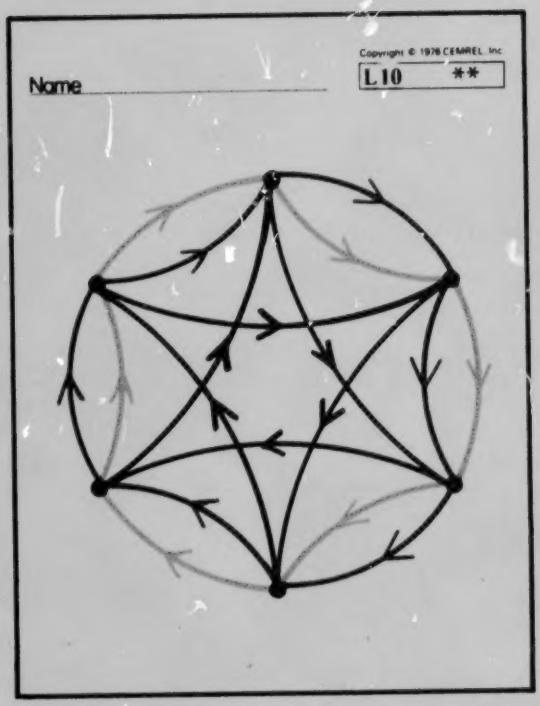


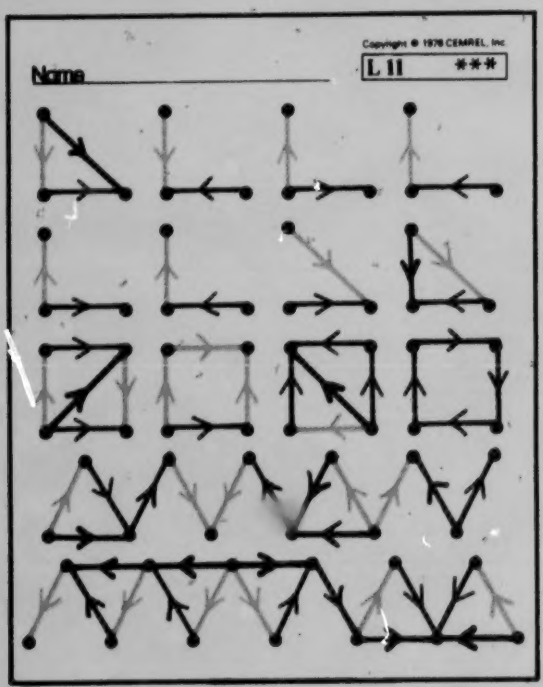
Use Worksheets L09, L10, L11 and let the students play the "yellow... then blue" game individually.

You can check Worksheet L09 easily because there should be exactly one red arrow in each of the first four "columns" of arrows on the left and no red arrows in the two "columns" of arrows on the right.

Worksheet L10 should have red arrows that form two triangles in the shape of a six-pointed star (if the arrows are drawn inside the circular figure).

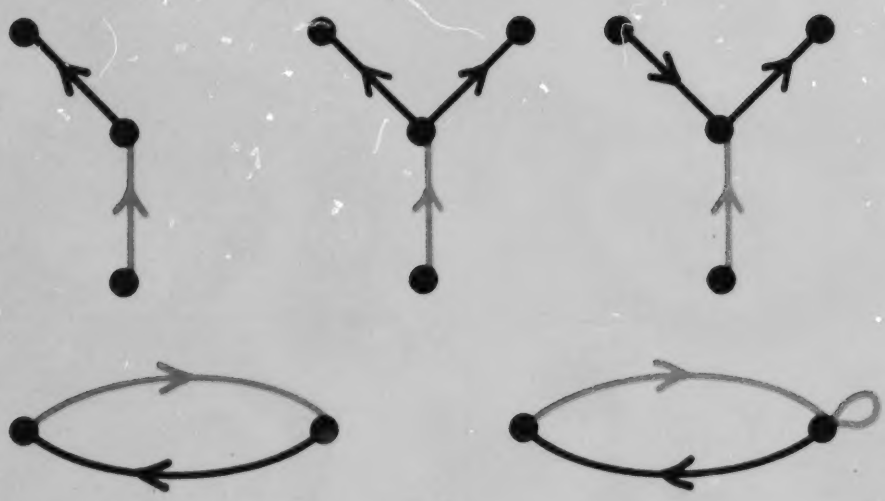

Copyright © 1978 CEMREL, Inc
Name
L 09 *


Copyright © 1978 CEMREL, Inc
Name
L 10 **

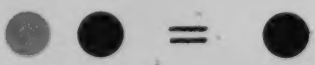

Copyright © 1978 CEMREL, Inc
Name
L 11 ***

3.2 Composition Games #2

Draw this arrow picture on the board:



and below it draw (with big colored dots):



T: Remember the "yellow...then blue" game; the game of finding all the red arrows?

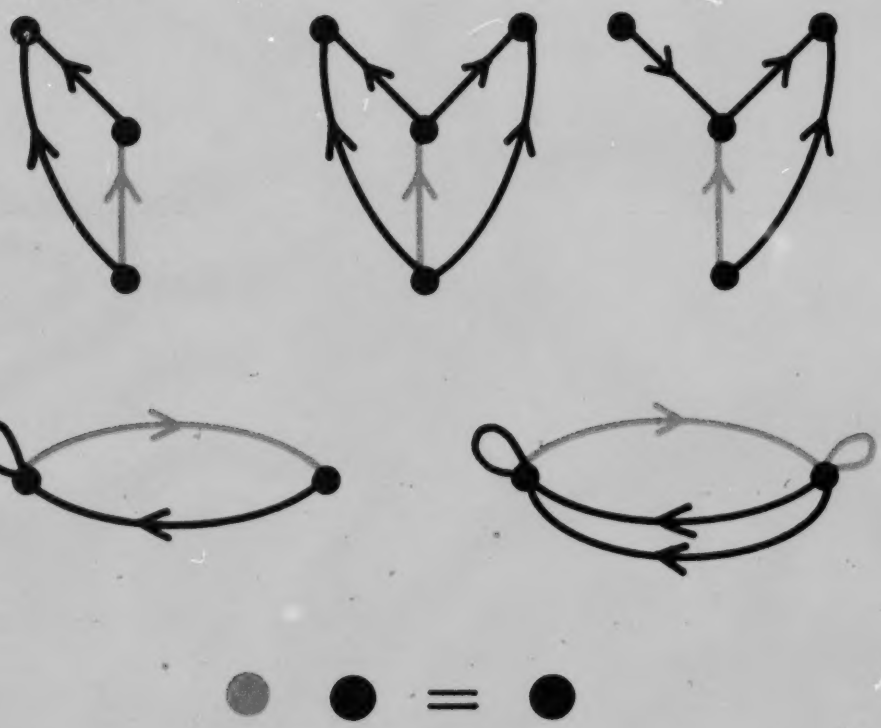
As you say this, point to the color rule that you drew below the arrow picture.

T: Who can find some red arrows?

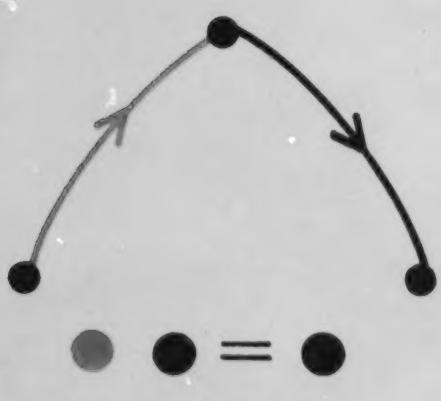
Let students, one by one, trace arrows on the picture. Draw the arrows yourself after they have been correctly traced. Help students who have

difficulty with this by holding their left forefinger at the dot where the yellow arrow starts and then helping them to trace the yellow arrow followed by the blue arrow until their right forefinger is at the dot where the blue arrow ends.

The drawing will finally look like this:



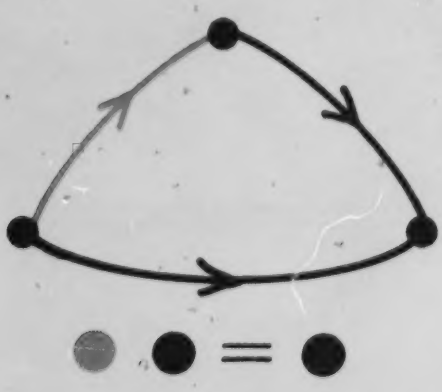
After this review of the composite game, draw a very large copy of this arrow picture on the board:



T: Can someone draw a red arrow?

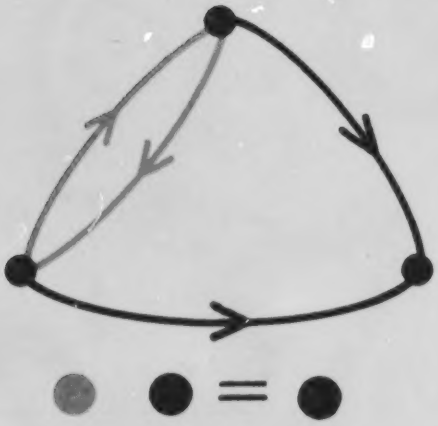
S: That's easy!

Let a student draw in the red arrow. Now the arrow picture will look like this:



T: I'm going to draw another yellow arrow. Now can you find more red arrows?

As you say this, draw a yellow arrow to make the picture look like this:



S: There are no more red arrows.

Let students check the picture to see that the new yellow arrow does not create any new red arrows.

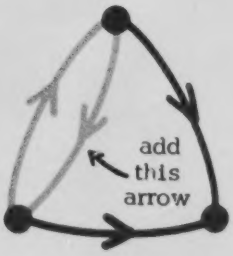
Then successively add arrows to the picture on the board, each time asking what new red arrows we get, and then drawing the red arrows. The arrows to be added, and the resulting new red arrows, are indicated in the following sequence of ten pictures:

6L

1

2

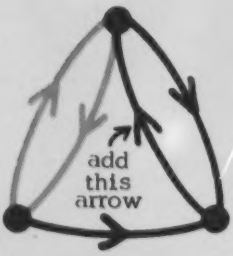
3



4

NO
MORE
RED
ARROWS

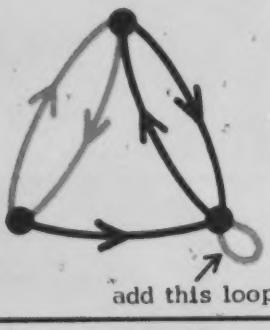
5



6

NO
MORE
RED
ARROWS

7



8

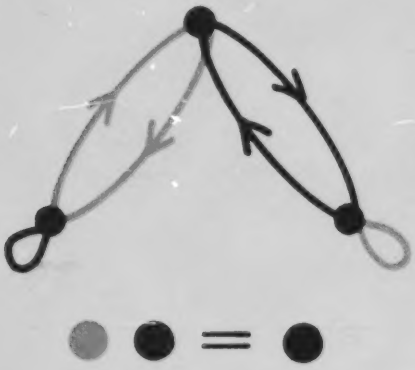
9

10

94

121

At the end of this experience, it will be interesting for the students to see a summary. This time draw a new arrow picture containing the same yellow and blue arrows but no red arrows:



T: What red arrows must we draw? What shape will the red arrows form?

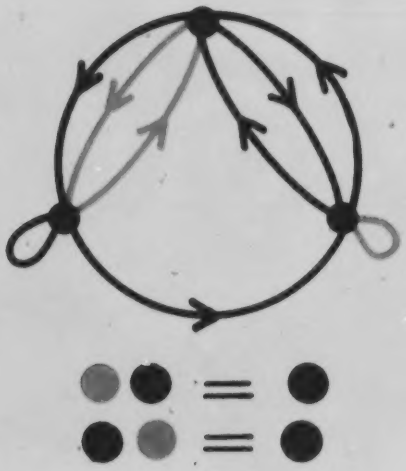
S: A circle.

Let students draw the red arrows. The picture will look like this:

S: The red arrows are chasing each other.

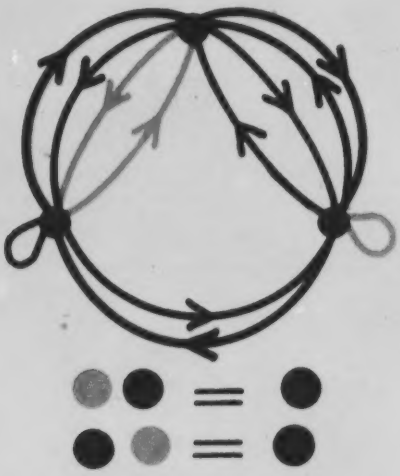
T: We will call it a red <u>cycle</u>.

Below the same picture, draw a new color rule and make no comments as the students look at it:



T: Who can draw some green arrows?

Let the students ponder this question and make some trials until they realize that "blue...then yellow" results in a green arrow. The figure eventually will look like this:



T: What do you notice?

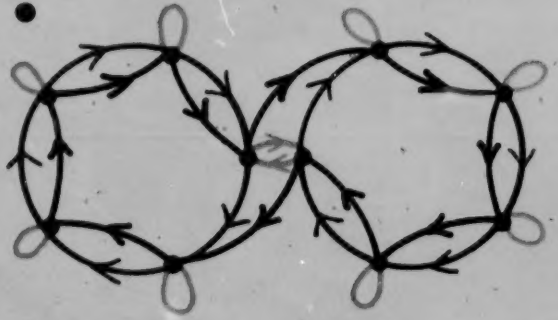
S: The green arrows go the other way from the red ones.

S: A green cycle!

For the remainder of the period, Worksheets L12, L13, L14 are available. Ask the students to draw all the red arrows in L12 and all the red and green arrows in L13 and L14.

Name

L12 *



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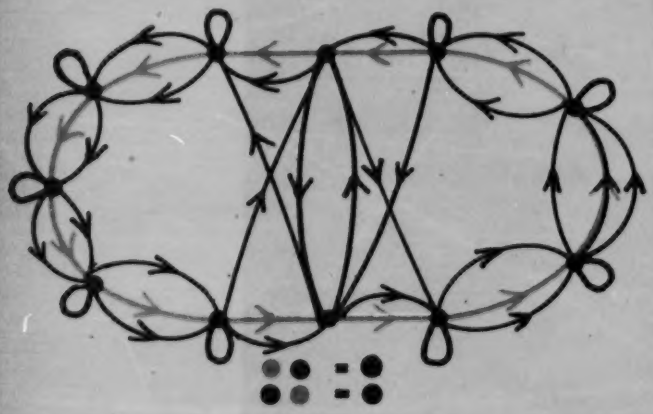
Name

L14 **



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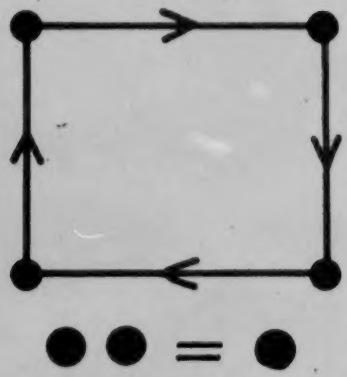
Name

L13 **

3.3 Composition Games #3

Draw on the board:



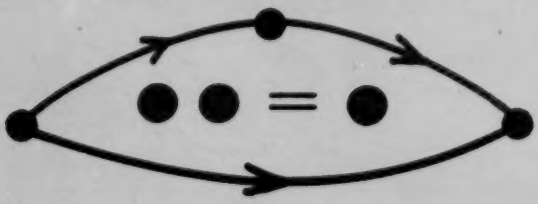
S: It's a blue cycle!

Point to the color rule below the arrow picture.

T: What could this mean?

S: Two blues equal one red.

Accept any answers of this sort. When you refer to the color rule you might say, "Blue followed by blue equals red." In order to make it clear what is meant, draw this arrow picture off to the side of the first picture you drew:

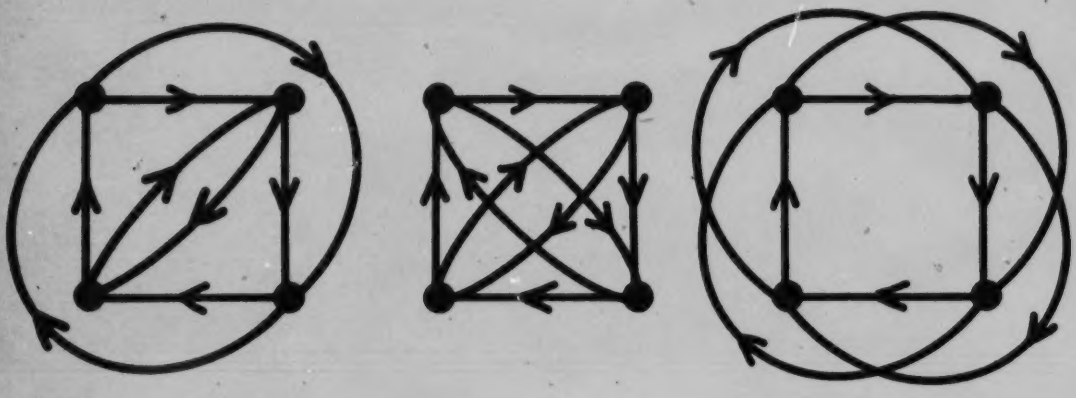


and again, with your forefingers, trace from the dot at which the first blue arrow starts to the dot where the second blue arrow ends.

T: Whenever a blue arrow is followed by a blue arrow, we have to draw a red arrow. Blue followed by blue equals red.

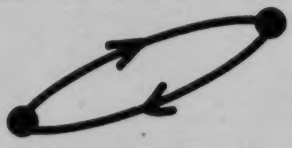
T: Now copy the arrow picture and draw all the red arrows you can.

As you walk around the room you might see a variety of correct responses, such as:

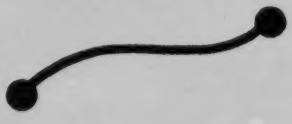


After answers are compared you can explain a simplifying notation.

T: We can make the drawings simpler. Whenever we have a "round trip" (draw):

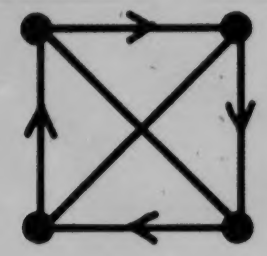


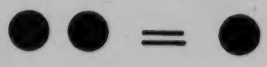
... we can draw this instead to make the picture simpler to read:



T: So here is my answer to the problem.

Draw red cords on the picture on the board so that it looks like this:

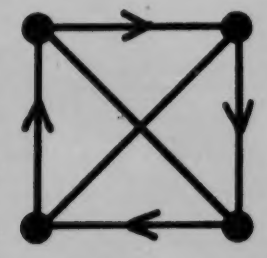


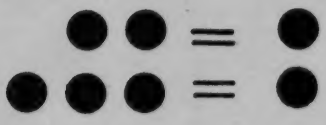


Explain how each red arrow is obtained, tracing the arrows and saying, rhythmically,

blue, blue, RED,
blue, blue, RED,
blue, blue, RED,
blue, blue, RED.

Now add a new color rule at the bottom of the picture, so that it looks like this:

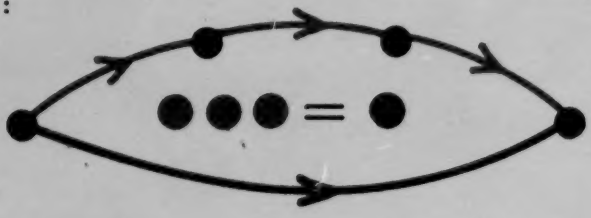




T: What does this mean? (Pointing to the new color rule.)

S: Three blues equal one green...

Again, accept any answer of this sort, and to the side of the picture on the board, draw:



T: Whenever there is a blue arrow followed by a blue arrow followed by a blue arrow, we have to draw a green arrow. Blue followed by blue followed by blue equals green.

S: Like elephants following each other in a circus.

Again you should trace the three blue arrows with your finger, saying, "Blue, blue, blue" ... (then trace the green arrow) ... "green".

Let the students take a few minutes to draw all the green arrows on their papers.

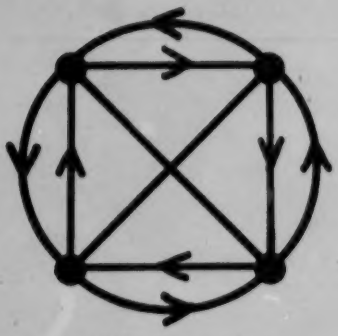
T: Tell me what the green arrows look like?

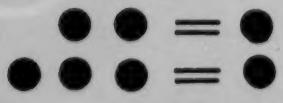
S: Another square.

S: A green cycle.

S: In the opposite direction from the blue cycle.

Draw the green arrows on the picture on the board:





Explain each green arrow by tracing the arrows and saying

"Blue, blue, blue, GREEN,
blue, blue, blue, GREEN, ..." etc.

T: What color rule do you think I will give you next?

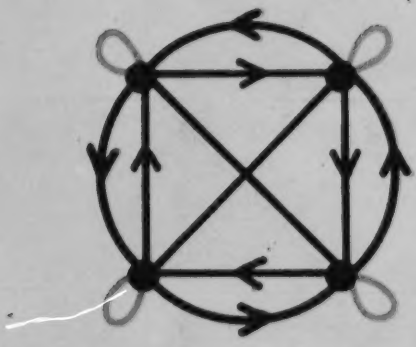
Generally, you will get the answer:

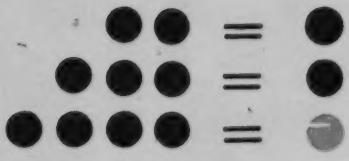
S: Four blue arrows!

Draw the color rule:

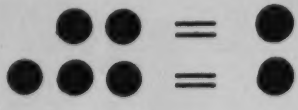
below the picture. Even before you go on with the lesson, some of your students will already have drawn yellow loops at the four dots on their papers. Then with your finger, trace the arrows, with the class probably chorusing with you,

blue, blue, blue, blue, YELLOW LOOP,
blue, blue, blue, blue, YELLOW LOOP, ... etc.

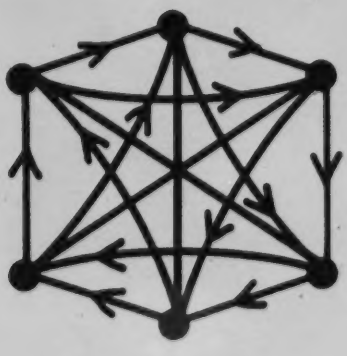




If there is time left at the end of the period, let the students draw their own picture starting with a blue cycle of six dots, using the color rules:

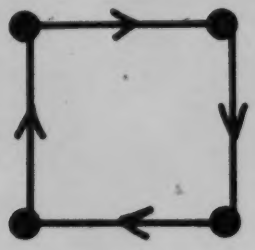


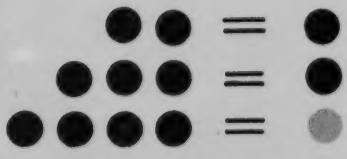
This is the resulting picture:



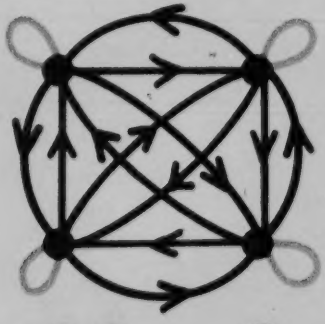
3.4 Composition Games #4

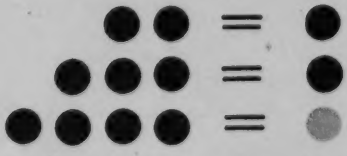
Review the last lesson on composition (Composition Games #3) by drawing this arrow picture on the board along with the three color rules:





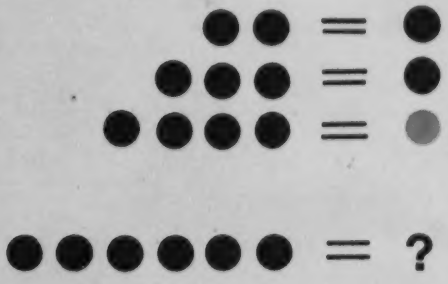
Let students quickly come to the board and draw all the red, green, and yellow arrows. (They will probably prefer to call the yellow arrows "loops".)





Again trace your finger along, for example, three blue arrows, saying "blue, blue, blue, GREEN", whenever a student has trouble at the board.

Now tentatively add another color rule:



T: What color?

Here there may be a great variety of opinions.

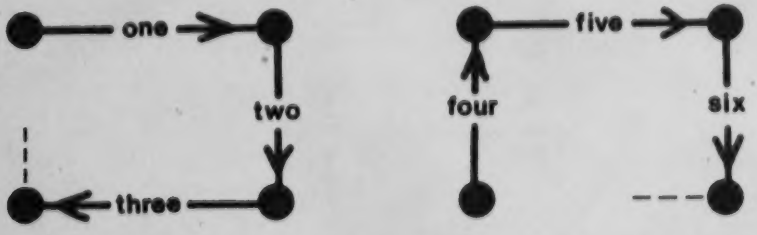
S: Red!

S: Two blues!

S: It's not possible, we have only four points.

S: You have to stop after four blue arrows.

You should be pleased if you get either or both of the first two responses, but also listen attentively to the last two. After some discussion, settle the question by placing your left forefinger on one of the dots and follow the arrows while the class counts...



T: We started here. (Tap the starting dot with your left forefinger.)
We ended here. (Tap the ending dot with your right forefinger.)
Then we have to draw an arrow from this dot... (Tap it.)
to this dot... (Tap it.)
But what arrow has already been drawn from this dot... (Tap it.)
to this dot? (Tap it.)

S: RED!

Then repeat the procedure at each of the other three dots, quickly but emphatically. Finally, ask a student to complete the color rule:

T: What will ELEVEN blue arrows make?

S: Three blues.

S: Green!

If any of these responses is made, ask the student to explain.

S: I guessed, but I can't explain it.

OR:

S: After four blue arrows you come back to the starting dot. After four more you come back again to the starting dot. That's eight. There are still three blues left: GREEN.

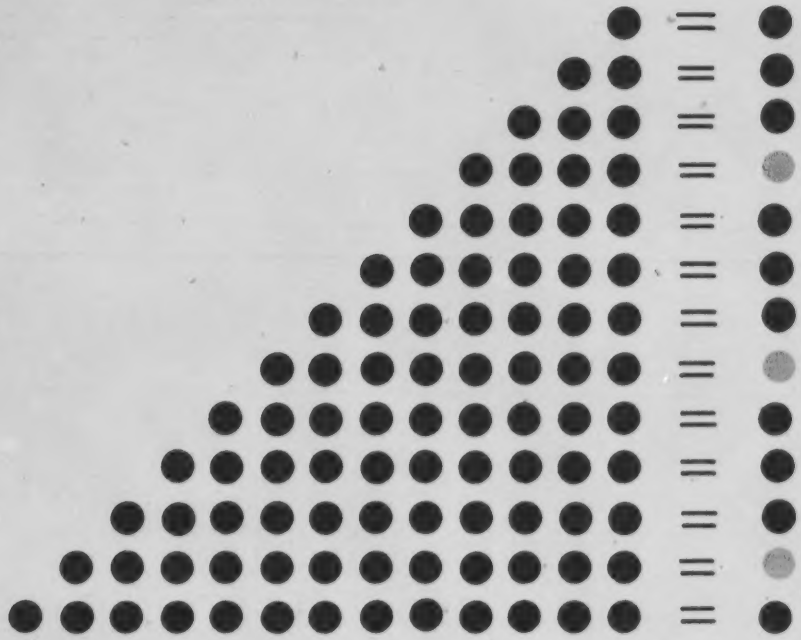
If no explanation is offered, go through the above explanation, again with the class counting as you move from a dot along the arrows. Then draw:

S: BLUE!

T:

S: GREEN!

Continue until the color rules under your drawing on the board look like this (or more, if the students want to go on):



T: Will it ever end?

S: No.

T: And there are just four colors we can get.

T: What color is 33 blues?

Let the students argue about this until it is decided that 33 blues equals blue.

T: 60 blues? (Answer: yellow)

T: 62 blues? (Answer: red)

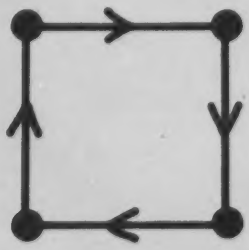
T: 100 blues (Answer: yellow)

If the class enjoys this, continue with other numbers of blues.

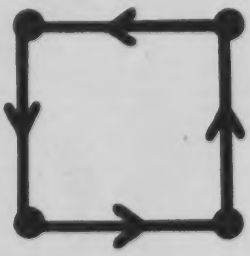
Point to the completed arrow picture.

T: Which color arrows forms one cycle?

After students discuss the answer to this question, draw:

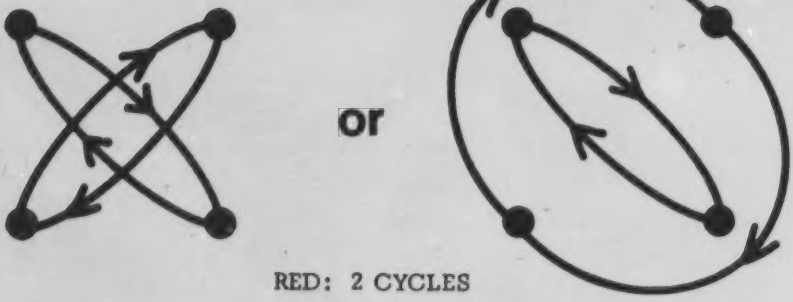


BLUE: 1 CYCLE



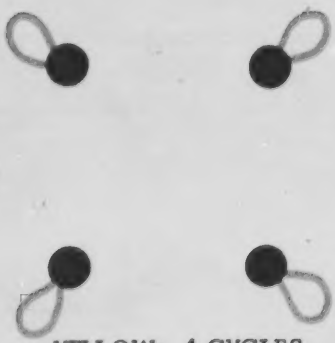
GREEN: 1 CYCLE

T: Which color forms two cycles?



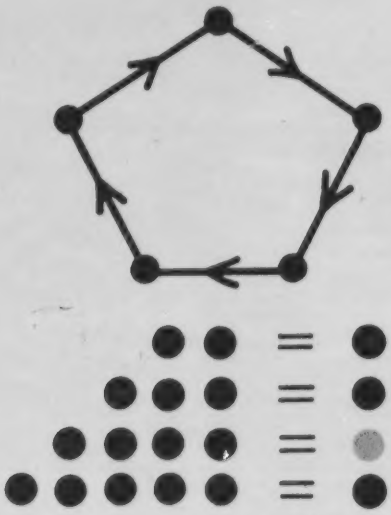
RED: 2 CYCLES

T: Which color forms four cycles?



YELLOW: 4 CYCLES

If there is time left at the end of the period, ask the students to copy the following arrow picture and color rules, and then draw all the arrows and give more color rules if they can:



Answer:

Geometry and Measurement

Comprehensive School Mathematics Program

Mathematics for the Intermediate Grades Part I

Geometry and Measurement

CEMREL, Inc.

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SUGGESTED SCHEDULE FOR TUESDAY'S AND THURSDAY'S LESSONS

WEEK	TUESDAYS	MATERIALS
1	1L L1.1: The String Game with A-Blocks p. 2	Teacher: Colored chalk; The String Game; 2 calculators (optional)* Student: None
2	2L L1.2: The String Game with Numbers: Multiples and Order p. 12	T: Colored chalk; The String Game S: None
3	3L L2.1: Logical Thinking #1 p. 52	T: Colored chalk; The String Game S: Worksheets L01, L02; pencils
4	4L L2.2: Logical Thinking #2 p. 61	T: Colored chalk; The String Game S: Pencils; paper; Worksheets L03, L04
5	4G G2.1: Circles #1 p. 39	T: Compass; straightedge; colored chalk S: Compass; 30 cm ruler; colored pencils
6	5G G2.2: Circles #2 p. 43	T: Compass; straightedge; colored chalk S: Compass; 30 cm ruler; colored pencils
7	6G G2.3: Circles #3 p. 47	T: Compass; straightedge; colored chalk S: Compass; 30 cm ruler; colored pencils
8	FREE DAY	
9	7G G1.4 Area Problems p. 29	T: Grid board; Flip-flop aid; colored chalk S: Worksheets G06, G07; colored pencils

SUGGESTED SCHEDULE FOR TUESDAY'S AND THURSDAY'S LESSONS

THURSDAYS	MATERIALS
1G G1.1: Comparisons of Lengths and Areas p. 3	T: Grid board; colored chalk S: 30 cm rulers; Worksheets G01, G02; colored pencils
2G G1.2: Introduction to Millimeters p. 12	T: None S: 30 cm rulers; Worksheets G03
3G G1.3: Square Regions p. 19	T: Four colored squares; grid board; colored chalk S: Colored pencils; Worksheets G04, G05
FREE DAY	
1P P1.1: Same or Different #1 p. 2	T: Marbles; colored chalk S: Marbles; pencils; paper
2P P1.2: Same or Different #2 p. 11	T: Marbles; colored chalk S: Pencils; paper
3P P1.3: Same or Different #3 p. 16	T: Colored chalk S: Pencils; Worksheets P01, P02, P03, P04, P05, P06, P07
4P P1.4: Same or Different #4 p. 22	T: Marbles; colored chalk S: Marbles; colored pencils; pencils; paper
5P P1.5: Same or Different #5 p. 30	T: Marbles; colored chalk S: Marbles; colored pencils; pencils; paper

SUGGESTED SCHEDULE FOR TUESDAY'S AND THURSDAY'S LESSONS

WEEK	TUESDAYS	MATERIALS
10	5L L3.1: Composition Games #1 p. 80	T: Colored chalk S: Colored pencils; Worksheets L09, L10, L11
11	6L L3.2: Composition Games #2 p. 90	T: Colored chalk S: Colored pencils; Worksheets L12, L13, L14
12	8L L3.3: Composition Games #3 p. 99	T: Colored chalk S: Colored pencils; paper
13	10L L3.4: Composition Games #4 p. 105	T: Colored chalk S: Colored pencils; paper; pencils
14	FREE DAY	
15	FREE DAY	
16	6P P2.1: "The Square Trap" #1 p. 36	T: "The Square Trap"; Minicomputer Set S: "The Square Trap"; Worksheet P08; spinner arrows; Minicomputer checkers; pencils
17	7P P2.2: "The Square Trap" #2 p. 38	T: "The Square Trap"; Minicomputer Set S: "The Square Trap"; Worksheet P08; spinner arrows; pencils; paper
18	8P P2.3: "The Square Trap" #3 p. 43	T: "The Square Trap"; Minicomputer Set S: "The Square Trap"; Worksheet P08; spinner arrows; pencils; paper

SUGGESTED SCHEDULE FOR TUESDAY'S AND THURSDAY'S LESSONS

THURSDAYS	MATERIALS
FREE DAY	
7L p. 67 L2.3: Logical Thinking #3	T: Colored chalk; The String Game S: None
9L p. 72 L2.4: Logical Thinking #4	T: Colored chalk; The String Game S: Worksheets L05, L06, L07, L08; pencils
FREE DAY	
8G p. 51 G2.4: Circles #4	T: Compass; straightedge; colored chalk S: Compass; 30 cm rulers; colored pencils
9G p. 59 G2.5: The Triangle Inequality	T: Compass; straightedge; colored chalk S: Compass; 30 cm rulers; colored pencils
11L p. 14 L1.3: The String Game with Numbers: Multiples, Divisors and Order #1	T: Colored chalk; The String Game S: Pencils; paper; colored pencils
12L p. 21 L1.4: The String Game with Numbers: Multiples, Divisors and Order #2	T: Colored chalk; The String Game S: Colored pencils; pencils; paper
13L p. 27 L1.5: The String Game with Numbers: Multiples, Divisors and Order #3	T: Colored chalk; The String Game S: Colored pencils; pencils; paper

SUGGESTED CHRONOLOGICAL ORDER OF "G" LESSONS

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GEOMETRY AND MEASUREMENT

INTRODUCTION

Geometry is useful and interesting, an integral part of mathematics. Geometric thinking permeates mathematics and relates to most of the applications of mathematics. Geometric concepts provide a vehicle for exploring the rich connections between arithmetic concepts (such as number and calculations) and physical concepts (such as length, area and shape). Even as children experience arithmetic notions in their daily lives, they encounter geometric notions. Common opinion recognizes that each child should begin developing a number sense early. There is no sound argument to postpone the development of geometric sense.

A benefit of early development of geometric thinking is that it provides alternative opportunities for "slow learners". Children develop number concepts at individual rates. Relative lateness of a child's numerical development often signifies a difficulty in remembering number facts. A common response to this problem is assignment of arithmetic drill which serves to confirm the diagnosis of slowness. However, numerical development is not an accurate indication of other development, in particular, of geometric development. Often the labels "slow" and "fast" are incorrect when alternative activities are presented.

The lessons of this strand present an opportunity to explore geometric notions informally. The activities are varied to meet many other areas of the program and are diffused over a long period of time. As a natural consequence of their involvement in the activities, the children will develop knowledge and skills. But the effects of such an informal program must be measured in their appearance in subsequent contexts. No goals for mastery can be posted here; none could be certified. The focus of this strand is experience.

1

MEASUREMENT

Consensus calls for measurement activities in the elementary curriculum, but with no agreement on the form or scope of these activities. Rather than stress mastery of formulas for area and perimeter or for comparison of standard units, the lessons of this chapter provide rich, open-ended experiences which relate measurement to other areas of the mathematics curriculum. The activities enable students to develop cognitive strategies and to build bridges among the strands of the program. Direct experience of the tools and concepts is central. Skills and mastery of measurement tasks will develop naturally.

Some lessons extend earlier experiences involving area and linear measure; others probe new directions. Area continues to be viewed as "the amount of paper covered". In one of the extending lessons, a series of linear configurations is constructed according to a pattern. Addition of a single line segment to each configuration produces a configuration of squares. While the linear configurations all have the same length, the areas of the successive square configurations decrease by a factor of one half.

In one of the probing lessons, the preservation of the area of a region under dissection and rearrangement of subregions is considered. After determining the area of a square placed at an angle to the grid lines, students find its area without counting half-squares, in particular, through dissection and rearrangement. Another lesson centers on a connection with the World of Numbers strand as students experiment in constructing squares of prescribed areas using four given squares. Students develop understanding of the relationship between the area of a region and the area of its subregions, both overlapping and non-overlapping.

That units of measurement are arbitrary, to be suited to convenience and convention, is implicit in the activities. Throughout the CSMP Elementary Program, standard units of measurement are those of the metric system (SI).

In previous lessons only the meter and the centimeter were used as units of length. In this chapter the millimeter is introduced through further work with linear measure.

NOTE ON GRIDS

In some of the lessons in Chapter 1, the children will require grid paper. Any grid paper will serve provided that the squares are not too small (less than $\frac{1}{2}$ cm). The ideal would be squares whose sides are 1 cm long.

In some of the lessons in Chapter 1, you will need a large grid board for class demonstrations. If there is none available in your classroom or school and your school is unable to purchase one, you can draw a square grid on your chalkboard. For this purpose you may find it useful to use the demonstration translator.* Squares with eight centimeter sides are ideal.

CAPSULE LESSON SUMMARIES

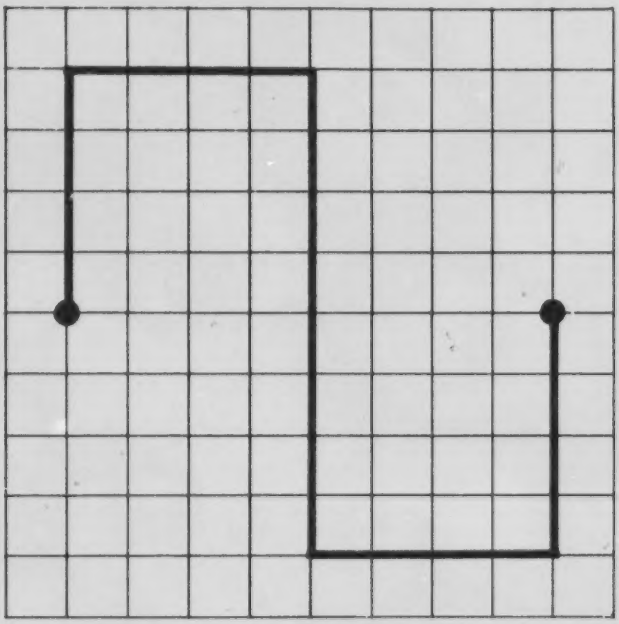
Section 1.1:	Calculate areas of patterns of squares and triangles to give practice in pattern recognition and in use of the properties of area. (Worksheets G01, G02)
Section 1.2:	Introduce the millimeter as a unit of length and give practice in measurement and in the use of the decimal point. (Worksheet G03)
Section 1.3:	Challenge: given four squares, construct regions of specified area. By attacking the challenge, consider related number concepts and properties of area. (Worksheets G04, G05)
Section 1.4:	Determine the area of a region through dissection and rearrangement of subregions. Present properties of area and develop efficient calculation skills. (Worksheets G06, G07)

* Since there is no other need for the demonstration translator this semester, it is not included with materials supplied by CSMP Mathematics for the Intermediate Grades, Part I. However, a demonstration translator is supplied with both Entry Module I and Part II materials. Each of you has one of these sets of materials for the other semester of this year.

1.1 Comparisons of Lengths and Areas

Distribute copies of Worksheet G01. Each student should have available some colored pencils and a ruler. Draw the following on your grid board and ask your students to copy it in the upper left-hand corner of their worksheet:

Figure 1



If necessary, give a verbal description of what you are doing as you draw it: "Up 4 cm, across 4 cm, down 8 cm, across 4 cm, up 4 cm."

Proceed similarly for the following three figures. The children should finish with one of these figures in each corner of their worksheet.

Figure 2

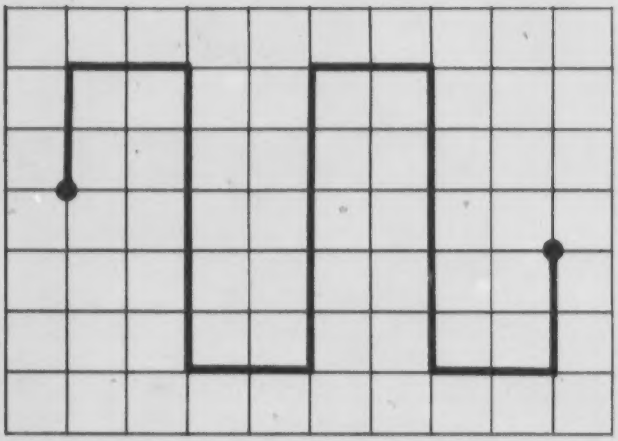


Figure 3

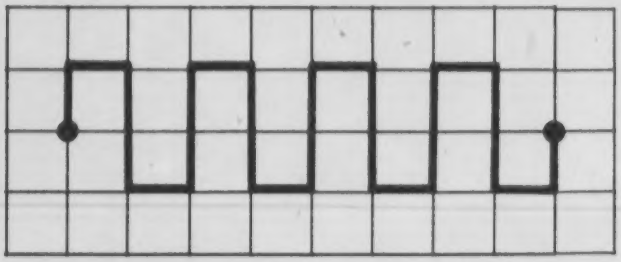
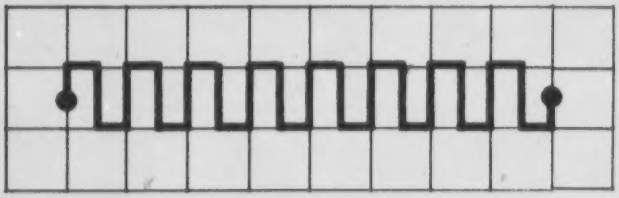


Figure 4



With Figure 4 emphasize that the blue zigzag starts and ends in the middle of a "block" and every other vertical segment goes down through the middle of a "block".

T: I want you to imagine that you have laid a piece of string along each of these blue zigzags, starting at the dot at one end and going to the dot at the other end. If you pulled those pieces of string out straight, which one would be the longest?

Children usually have a fairly good intuition about this. Probably most of your class will agree that all the strings would be the same length.

T: How would you convince someone that the strings are the same length if that person thought they weren't?

Reshape and draw together the responses to form a convincing argument. Here's how the discussion might go:

S: The one that looks the biggest (Figure 1) is more spread apart, and the small one (Figure 4) wiggles up and down more.

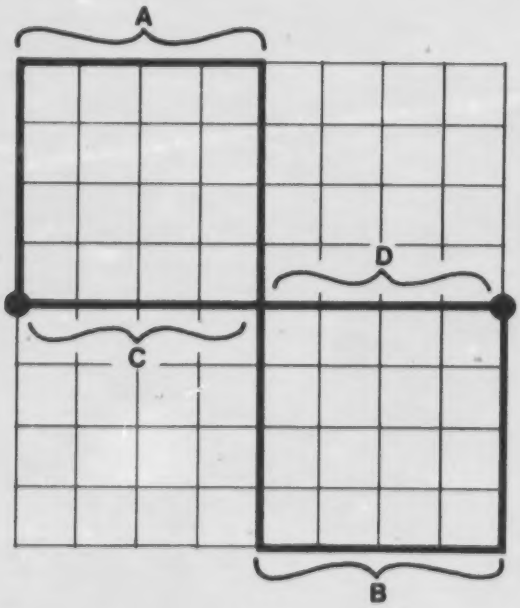
S: The distance between the two end dots is 8 cm for all the pictures.

S: I counted the blocks. It comes to 24.

S: They look the same upside down as right side up.

T: It's very important that the distance between the two end dots is the same. Look at this.

Draw a red line between the end dots in Figure 1 as follows:



T: If I draw a red line like this between the end dots in each of the four pictures, then all the red lines are the same length. Now this blue piece (A) is the same length as this red piece (C), and this blue piece (B) is the same length as this red piece (D). That means that together these two blue pieces are the same length as the distance between the two end dots. Is it like that with the other three pictures?

It is easy to see that the other pictures are similar.

T: We have found that if you add up the lengths of all the horizontal* pieces you get the same answer in all four pictures. What about the vertical* pieces?

S: In this picture (Figure 1) you've got one piece that's eight centimeters long and two pieces that are four centimeters long. That's sixteen centimeters altogether.

Write on the board:

$$(1 \times 8) + (2 \times 4) = 16.$$

Proceed similarly for the other figures, adding to your list of calculations on the board as you go:

$$(3 \times 4) + (2 \times 2) = 16$$

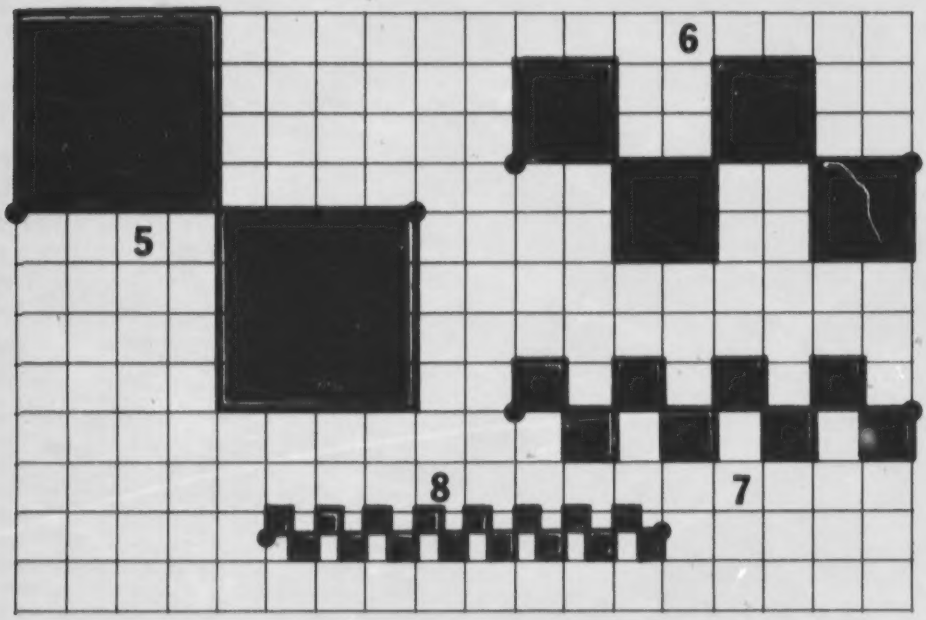
$$(7 \times 2) + (2 \times 1) = 16$$

$$(15 \times 1) + (2 \times \tfrac{1}{2}) = 16.$$

* Explain the meaning of this word if necessary.

T: If you add up the lengths of all the vertical pieces you get the same answer in all four pictures. Horizontal — the same, and vertical — the same; all four lengths must be the same. Notice that it's just as someone said at the beginning: in the pictures where the pieces are smaller, there are more pieces and that makes it work out to be the same.

Now, ask the students to draw a blue line between the end dots in each of their four figures and color the resulting boxes in red as follows:



T: Which of these figures has the largest region colored red? Or, do you think all the areas are the same, as the lengths were?

Once again, you will find that children usually give the correct answer: the areas are <u>not</u> equal and Figure 5 has the largest area.

T: How do you know the areas are different?

S: It takes four squares this size (i.e., of one of the four square regions in Figure 6) to fill this square (one of the two square regions in Figure 5). That means all the parts of this shape (Figure 6) would fit inside this square which is only a half of this shape (Figure 5).

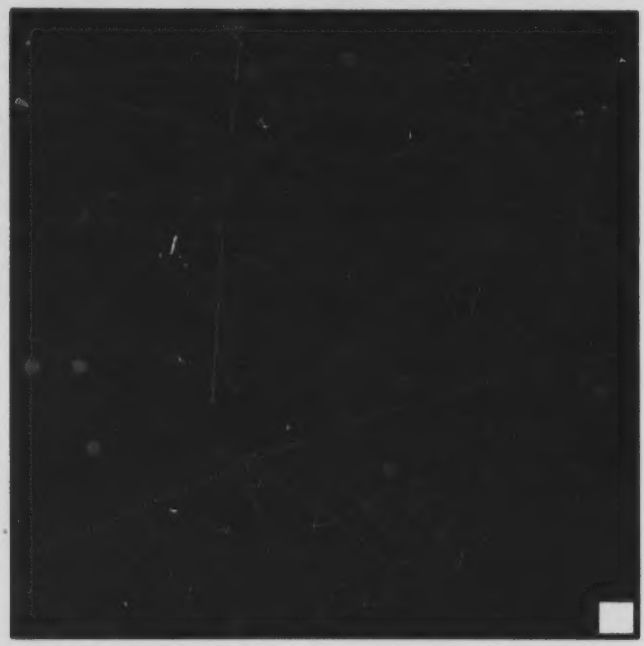
A similar argument shows that the area of Figure 7 is a half of that of Figure 6 and the area of Figure 8 is a half of that of Figure 7.

Distribute copies of Worksheet G02. Ask the children to color three parts of the big square as shown in this reduced copy:

8

T: Now divide the uncolored part of the big square into four and color three of those parts the same way.

Continue in this way until the children have colored the following picture:



T: Do you see how it would go on? If our colored pencils were sharp enough we could cut more white squares into four and color two parts red and one part blue. Do you think there would be more of the big squares colored red or blue?

S: More would be colored red.

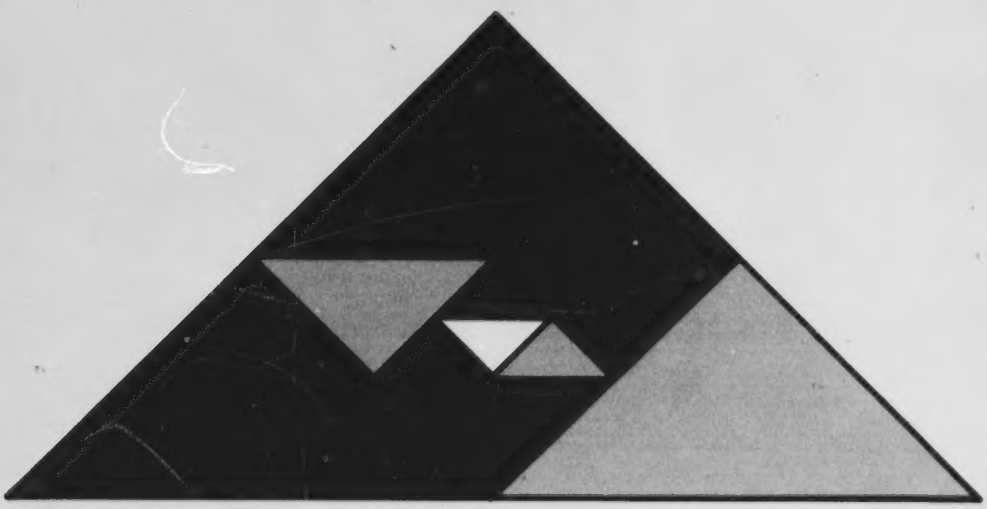
T: How much more?

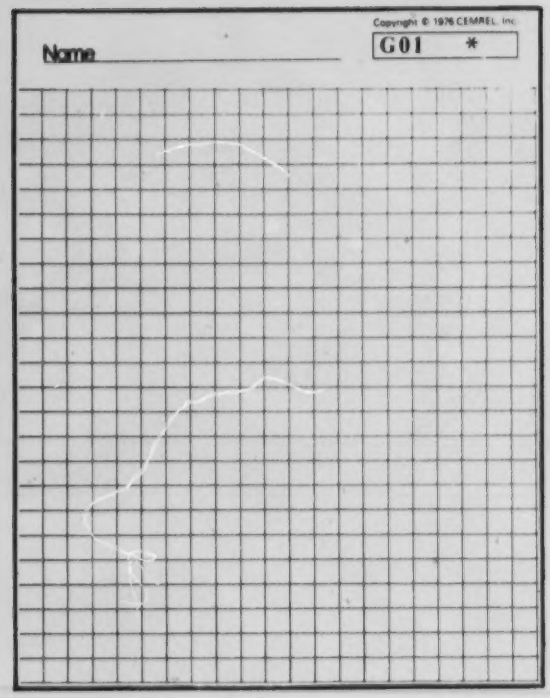
S: Twice as much red as blue. Everytime you color a blue square you color two red squares.

T: How big a part of the big square would end up being colored blue?

S: About one-third. You always color one out of three parts blue with just a tiny bit of the big square not colored.

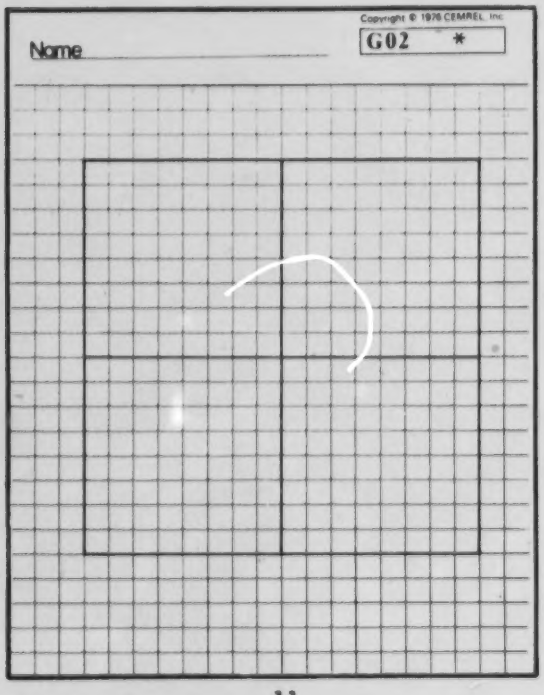
If you do not obtain this answer in your class and the children are frustrated by the question, don't push the subject any further. However, if the children seem to enjoy the activity and time allows, suggest that they try a similar drawing with a triangle. An intermediate stage in the procedure is shown in the following diagram. Each new stage is realized by joining the midpoints of the sides of the uncolored triangle and coloring three of the four parts. If this process is continued about one-third of the triangle will be colored in each color.





G01

See pages 1-7 for a description of how this worksheet is used in the lesson.



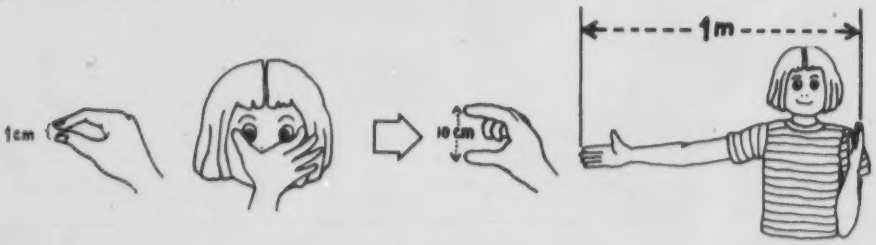
G02

See pages 8-10 for a description of how this worksheet is used in the lesson.

1.2 Introduction to Millimeters

Begin the lesson by reviewing what the children have learned about meters and centimeters, emphasizing the following points:

- approximations of 1 cm, 10 cm, 1 m in terms of body measurement.



- abbreviation of centimeter by "cm" and meter by "m".
- 100 cm = 1 m.

Distribute 30-centimeter rulers.

T: Look at your rulers. Which is the side with the centimeters?

S: The one that's numbered up to 30.

T: Look closely at that side of the ruler and you will see some little spaces marked off between the lines that mark the centimeter. How many of those little spaces are there in each centimeter?

S: Ten.

T: The length of each of those little spaces is called a "millimeter".

Write "MILLIMETER" on the board, and then add to the review material written on the board. Write:

10 millimeters = 1 centimeter

next to

$$100\text{ cm} = 1\text{ m}$$

T: Just as in the case of centimeters and meters, there's a shorthand way of writing "millimeter".

Write "mm" on the board below "MILLIMETER" and add

$$10\text{ mm} = 1\text{ cm}$$

to the list of equivalences.

Distribute Worksheet G03.

T: I would like you all to measure how long the blue piece of line is.*

S: 18 cm.

S: $17\frac{1}{2}$ cm.

S: I got 17 cm and 8 mm.

T: I also got 17 cm and 8 mm, so everyone should recheck their measurement to make sure that this is the correct length. This is the way it looks when you're measuring it.

---

* A common measuring mistake is made by placing the mark at 1 on the ruler on one end of what is being measured instead of the mark at 0. Those who do this will get answers 1 cm too large, so you can readily identify them. All measurements are to be taken to the nearest millimeter.

Draw the following enlargement:

T: Here's 17 cm. (Point) Then you count the millimeters until you reach the end of the blue piece of line. 1, 2, 3, ..., 8. So it's 17 cm and 8 mm. Rather than writing out "17 cm 8 mm" we sometimes write this in the same way as we would if it were 17 dollars and eight dimes.

Elicit from the children that this amount of money could be written as

$17.80 or $17.8

In the same way, 17 cm 8 mm could be written as

17.80 cm or 17.8 cm

T: What's the closest length to 17.8 cm that is a whole number of centimeters?

S: 18 cm.

T: Is that longer or shorter than 17.8 cm?

S: Longer.

T: How much longer?

S: 2 mm.

T: What's the nearest whole number of centimeters that is shorter than 17 cm and 8 mm?

S: 17 cm.

T: How much shorter is that?

S: 8 mm.

T: Turn your worksheets over. On the back of the worksheet draw a piece of a line which is longer than 11 cm and shorter than 12 cm.

S: Does it have to be a whole number?

T: Can it be a whole number?

S: No.

S: I've drawn one that's $11\frac{1}{2}$ cm long.

T: That's the same as 11 cm and how many millimeters?

S: Five.

T: What's another way of writing that using the decimal point?

S: "11.50 cm".

S: "11.5 cm".

T: Turn your worksheets back to the front side again. You'll find there two red pieces of lines put together to look like a "7". Measure those two pieces. Measure the top part first, and the slanting part second.

The correct measurements in this case are 10.7 cm and 18.4 cm respectively.

T: How can we find the total length of the red bent line?

S: Add them up.

Let the children do this addition individually, and then go through the addition on the board for the benefit of those who had difficulty. Of course, the total length is 29.1 cm.

T: How many millimeters is that altogether?

S: 291. 29 cm is 10 x 29 mm = 290 mm, and then there's one more millimeter.

T: Now measure the pieces of the black zigzag.

These lengths are 11 cm, 7.5 cm and 6.3 cm. In order for the students to be able to add these up using the standard algorithm, you may have to point out that "11 cm" can also be written "11.0 cm". The total length of the black zigzag is 24.8 cm.

T: Turn your worksheets over again.

Write on the board:

Draw a zigzag

which has three pieces

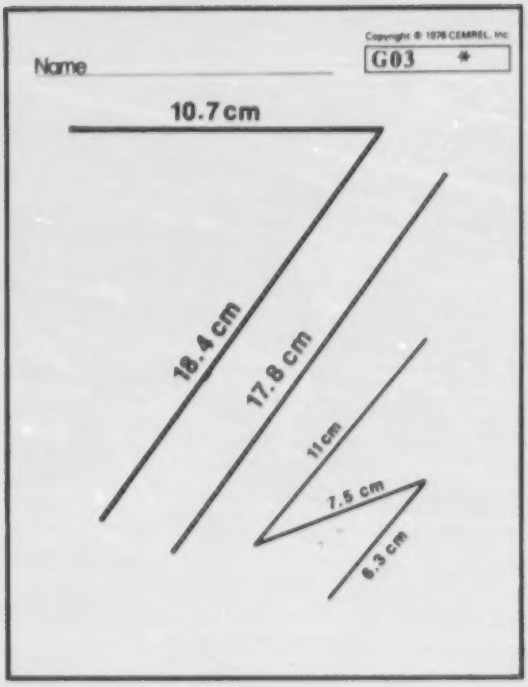
and which is exactly 56.7 cm long altogether.

Let the students work on their own, while you walk around correcting any gross errors. When most students have finished, ask for solutions. Show particular interest in those which minimize the amount of calculation necessary. For example,

Draw one segment 30 cm long,

a second segment 20 cm long,

and a third segment 6.7 cm long.


Copyright © 1976 CEMREL, Inc.
Name
G03 *
10.7 cm
18.4 cm
17.8 cm
11 cm
7.5 cm
6.3 cm

1.3 Square Regions

Hold up a copy of Worksheet G04.

T: Do you remember how long the side of one of the little squares on this paper is?

S: One centimeter.

T: Right. But as you notice, they're a little too small for you to be able to see clearly what I'm showing you. So today when I'm showing you anything at the front of the class you should pretend that I'm using "enlarged" centimeters.

Hold up the yellow square, non-grid side out.

T: I want you to imagine that each side of this square is 4 cm long. What is the area of the yellow region?

S: 16 cm.

T: Don't forget that we measure area in square centimeters. You're correct, the area is 16 cm^2. How could you explain why that is so to someone who didn't know?

S: $4 \times 4 = 16$.

S: You've got a row of four centimeter squares across the top, then another row of four, then another and another. Four rows of four makes sixteen.

If no one offers this last explanation, do so yourself. In any case, as soon as this explanation has been given, turn over your yellow square to reveal

the grid. Point out the four rows each consisting of four centimeter squares.

Deal with each of the other three squares in a similar way.

Hold the yellow square flat against the chalkboard, grid side facing out.

T: What area of the board am I covering?

S: 16 cm^2.

Hold the red square flat against the board, grid side facing out, adjacent to the yellow square.

T: Now what area of the board am I covering?

S: 25 cm^2. You covered up 9 cm^2 more.

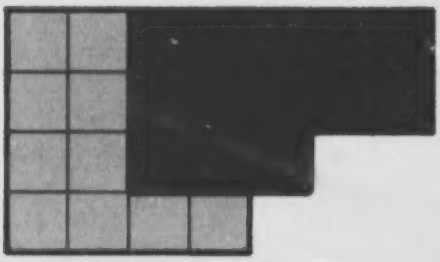
Next, add the green and the blue squares.

S: Now you're covering 30 cm^2.

T: Can I cover any more of the board with these four squares?

S: No.

Hold the squares flat against the board, grid side facing out, overlapping as in this configuration.



T: What area of the board am I covering now?

S: 23 cm^2.

T: Very good. There's 16 cm^2 here, then 3 cm^2 of this square sticking out, and finally all 4 cm^2 of this square. What's the smallest area I can cover if I have to use all four squares?

S: 16 cm^2.

T: How could I do that?

S: Just pile them on top of one another.

Write on the board:

16 30

T: The smallest area I can cover is 16 cm^2 and the largest is 30 cm^2. What do you think my next question is going to be?

S: Can you cover all the areas in between?

T: Yes, but I'll tell you exactly what that means. I'm going to insist that when you overlap two squares you do it along the lines. Like this

not like this

or this.

Distribute copies of Worksheet G04 and tell the students that they are to draw ways of arranging the four squares to construct regions with areas between 16 cm^2 and 30 cm^2.

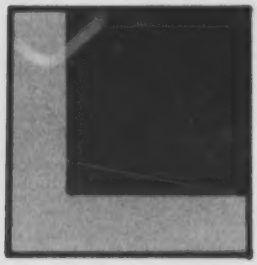
Explain the method of recording the positions of the squares by drawing their edges as we have done in the figures on the preceeding pages. It is possible to cover all whole number areas between 16 cm^2 and 30 cm^2 in this way. A complete set of configurations is at the end of this section. In all cases, there is more than one solution.

Be sure to maintain interest in the problem. For example, after five minutes ask if anyone has found a way to cover 17 cm^2. Probably someone will have done so. Send that person to the grid board to demonstrate. After a few more minutes do the same for 18 cm^2, and so on.

As soon as someone has found ways of getting all the intervening areas, announce that fact so that the other students know that it is possible. Then on to the next activity, suggesting that knowing it to be possible they can work on it in their spare time.

Hold up the yellow and the red squares, plain side facing out.

T: If I put the red square completely on top of the yellow square like this



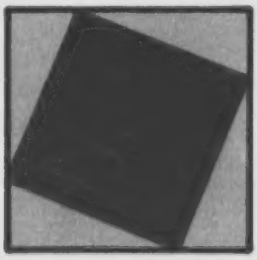
what area of the yellow square will be left showing?

S: 7 cm^2.

T: How do you know?

S: You have four yellow squares down that side and three more across the bottom.

Rotate the red square, still keeping it on top of the yellow square.



T: What area of the yellow square is showing now?

Your students will probably be confused and not know how to answer this question.

T: What area of the yellow square have I covered up?

S: 9 cm^2, the area of the red square.

T: So how much of the area of the yellow square is left showing?

S: Still 7 cm^2. 16 - 9 = 7. You just turned the red square around.

T: Do you think it's possible for me to lay these two squares flat against the board so that 16 cm^2 of yellow is showing?

Probably someone will suggest either laying the yellow square on top of the red one, or else laying them down side by side. Either of these solutions is correct.

T: Can I lay them down so that more than 16 cm^2 of yellow is showing?

S: No.

T: What is the smallest area of yellow I can have showing when I lay these two squares against the board?

S: 7 cm^2. You did it just now.

T: Do you think we can get all the whole numbers between 7 and 16?

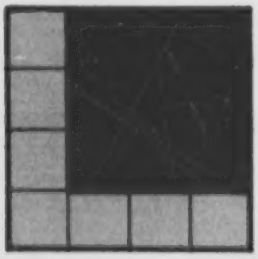
Distribute copies of Worksheet G05 and let the children work on this problem individually. They should have no difficulty finding that 7, 10, 12, 13, 14, 15, 16 are the only possibilities.

If you feel that some of your students are capable of extending this activity, announce:

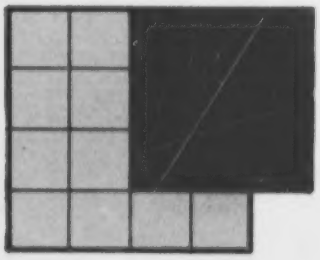
T: For this problem I'm going to allow you to break the rule that says you must overlap the squares on the lines. Then you can find the three missing areas (8 cm^2, 9 cm^2, 11 cm^2).

A complete set of configurations is at the end of this section.

Before ending the lesson, give one of the students' solutions for 8 cm^2, 9 cm^2 and 11 cm^2 or show how they could be constructed as follows:



7 cm^2



10 cm^2

So 8 cm^2 must be somewhere in between. You need one more square centimeter after 7 cm^2. But the edge of the red square covers 3 cm; that means you only have to move it a third of a square across, since $\frac{1}{3} \times 3 = 1$.

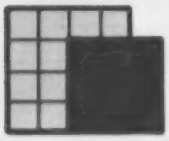
9 cm^2 and 11 cm^2 can be found similarly.

3G

16

17

18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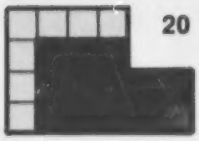
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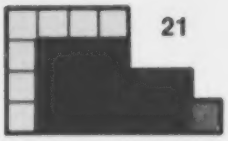
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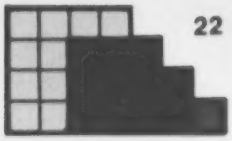
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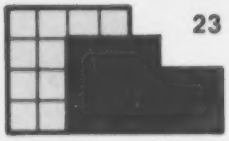
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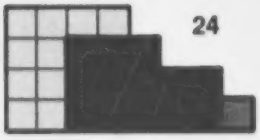
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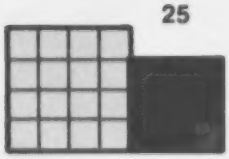
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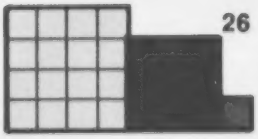
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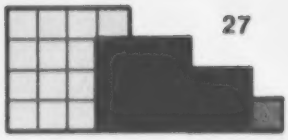
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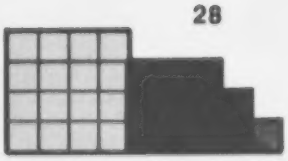
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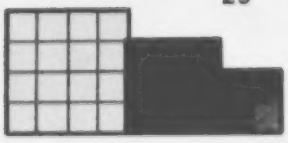
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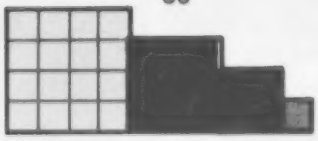
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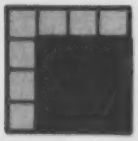
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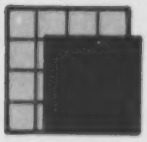
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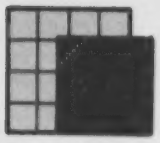
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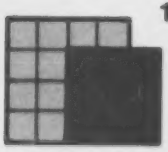
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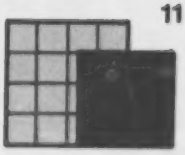
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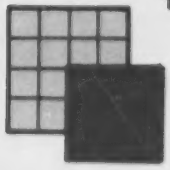
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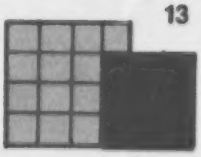
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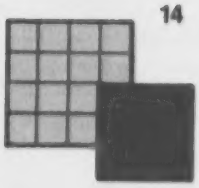
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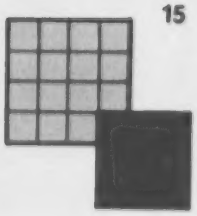
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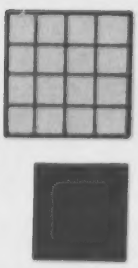
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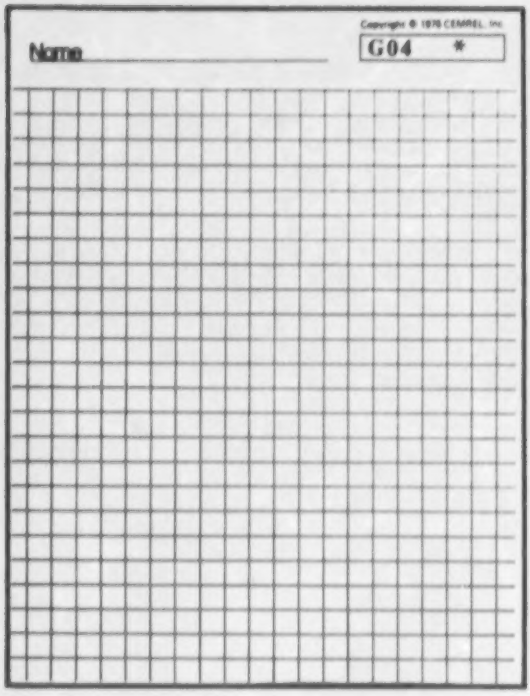
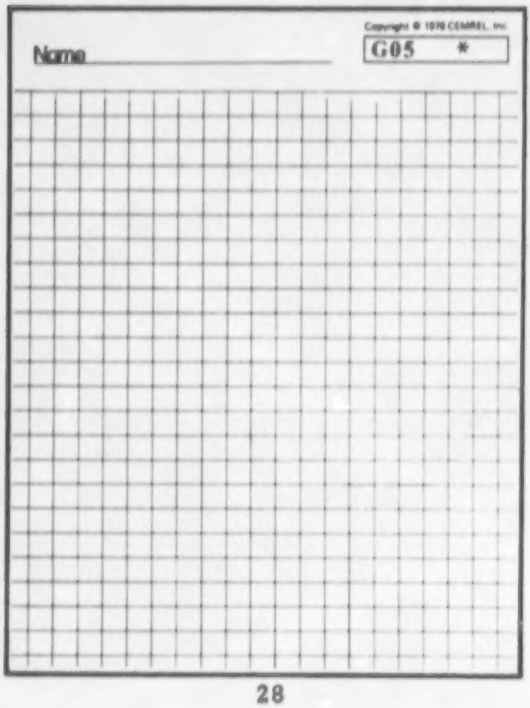


G04

Answer key for G04
is on page 26.



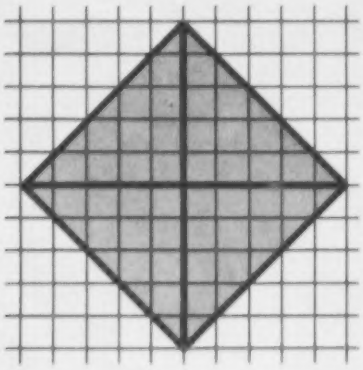
G05

Answer key for G05
is on page 27.

1.4 Area Problems

Exercise 1:

Draw the following on the board. Draw the diagonals first, each 10 cm long. Use a straightedge to draw the sides. Shade in the surface roughly with some colored chalk.



T: These two lines that go from corner to corner (indicate the diagonals) are called the "diagonals" of this shape. I want you to pretend that the squares of this grid board have sides that are 1 cm long. Then each of the diagonals of the shape I have drawn is 10 cm long, and they cross each other exactly at each other's midpoint. In other words, it's 5 cm from where they cross to each of the four corners. What's the name of the shape I have drawn?

S: Diamond ... Kite ... Square.

If you do not receive this last answer, ask the class if it is a square and then show that it is by checking that

a) each corner is square (use the corner of a piece of paper);

and

b) the length of each side is the same (use a compass to lay off the length of one of the sides against the other three).

T: What's the area of this square? How can we find out?

S: Count the squares.

T: That's a good idea, but what about the part-squares?

S: They're halves, so two of them make a whole.

Count the whole squares first, and then count the halves in pairs to find that the area is 50 cm^2.

T: Is there another way we could find the area without having to worry about half-squares?

Discuss suggestions that are given. Then, if it has not already been suggested, raise the following possibility.

T: Sometimes if you cut a shape up into pieces and rearrange the pieces you can make a new shape whose area is easy to find. And then, because the new shape will have been made from the pieces of the original shape, it will have the same area as the original shape. Can you see a way of cutting this square into pieces so that the pieces can be rearranged to form a shape whose area is easy to find?

S: Cut out the top two triangles and put them next to the bottom two triangles. That'll make two squares, side by side.

Demonstrate this, one upper triangle at a time, using the flip-flop visual aid:

T: What is the length of each side of these two squares?

S: 5 cm.

T: What is the area of each of the two squares?

S: 25 cm^2.

T: What is the area of the whole new shape?

S: 50 cm^2.

T: So we get the same answer both ways.

Exercise 2:

Distribute one copy of Worksheet G06 to each child.

T: What shape is drawn on the worksheet?

S: A square.

T: How long is each of its diagonals?

S: 20 cm.

T: How far is it from the place where the diagonals cross to each of the corners of the square?

S: 10 cm.

T: What is the area of the square?

S: 200 cm^2.

T: How did you find that?

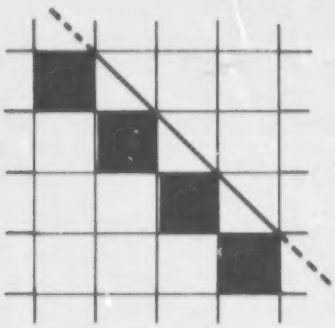
S: Four of these squares we had just now will fit into it, and 4 x 50 = 200.

Show how four of the 50 cm^2 square fit into this square for the benefit of those who hadn't noticed this fact. (Draw the segments from the center of the square to the midpoints of each of the four sides of the square.)

T: Now there's a task I want you to perform and I'm going to give you four instructions, so listen carefully. Here they are:

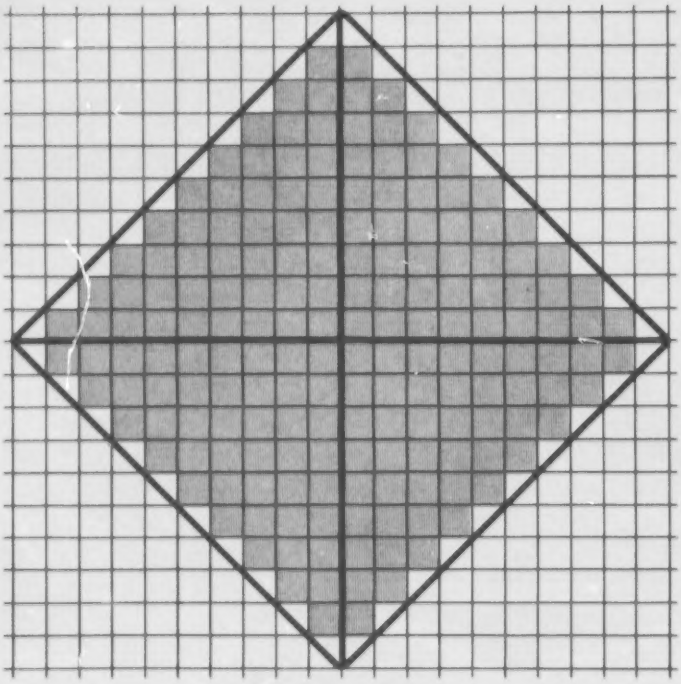
1. I want you to draw a shape.
2. The shape must be completely <u>inside</u> this big square on your worksheet.
3. The shape must be made only of whole squares — no half-squares, or other parts of squares.
4. The shape must have as big an area as possible.

Let's start work together. The shape must have as big an area as possible, so that means we have to stay as close to the edge of the big square as we can while using only whole squares. So we can include squares like this:



Go on working by yourselves.

Here's the complete solution to the problem:



T: What do you think I'm going to ask you next?

S: What is the area of the shape we just colored?

T: Right. What is it? How can we find it without having to count all the squares?

Discuss the suggestions of the class; go through the following two methods in detail (you may have to suggest them yourself):

Method 1: The big square has area 200 cm^2. You just have to take out the staircases. There are four staircases, each with the same area: $10 \times \frac{1}{2}\ cm^2 = 5\ cm^2$. Hence the colored area = $200 - (4 \times 5)\ cm^2 = 180\ cm^2$.

Method 2: Think of the shape as being made up of strips. It's symmetric about each diagonal, and the area of each half is

$$(2 + 4 + 6 + \ldots + 16 + 18)\ \text{cm}^2.$$

Let the children calculate this sum individually, and then, if no one suggests it, use the following method to check their work.

$$2 + 4 + 6 + \ldots + 14 + 16 + 18$$

$$= (4 \times 20) + 10$$

$$= 90$$

Hence the whole area = $2 \times 90\ \text{cm}^2 = 180\ \text{cm}^2$.

Exercise 3:

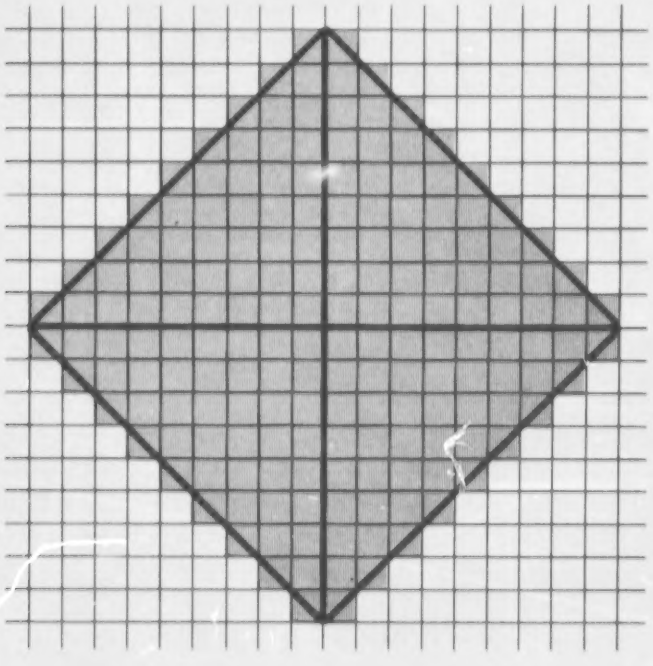
If time allows, ask the children to turn over their worksheets to Worksheet G07. Let them measure the lengths of the diagonals and find that this square is smaller than the one on the other side.

Set a task similar to the one in Exercise 2.

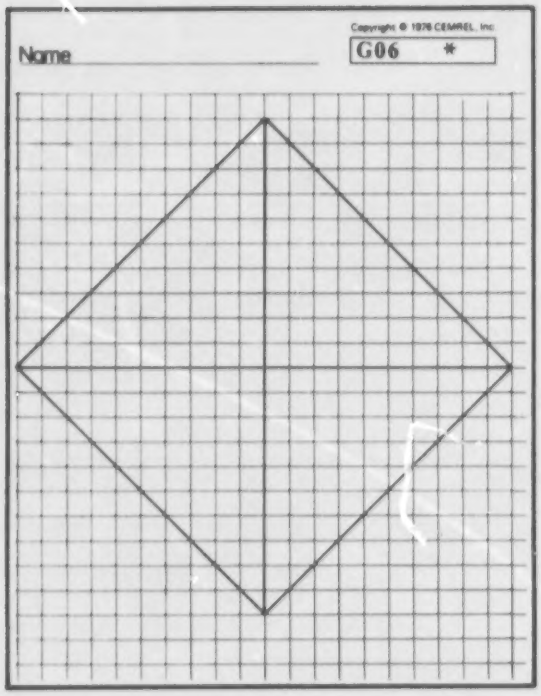
T:
1. I want you to draw a shape.
2. The big square on the worksheet must be completely inside this shape.
3. The shape must be made only of whole squares — no half-squares, or other parts of squares.
4. The shape must have as small an area as possible.

T: Do you think this shape will have a smaller, larger or the same area as the one we drew a moment ago?

There will be mixed reactions to this question. Leave it open and let the children work individually. Here is the complete solution:

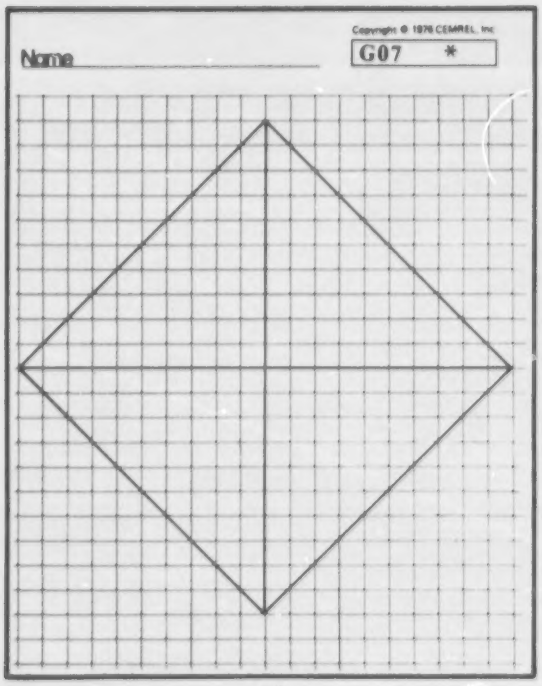


Work through two methods of calculating the area of this shape in much the same way as in Exercise 2 (you might need to use the flip-flop device again in order to help you calculate the area of the basic square). The area is 180 cm^2 again; indeed, it's exactly the same shape as the one in Exercise 2.



G06

See page 33 for the solution to this problem.



G07

See page 35 for the solution to this problem.

2

INTRODUCTION TO CIRCLES

Our geometry so far has been concerned with points and lines. The circle is another fundamental figure. The lessons of this chapter provide an introduction to the circle and its relations with the point and the line. At the most basic level, circles appeal through their symmetry and their contrast with rectilinear figures. Study of circles contributes to the students' developing aesthetic sense. At the immediate level, circles are necessary to the geometric structure. Study of circles contributes to the students' developing geometric intuition and knowledge. Again our focus is on experience with tools and concepts and not on mastery of facts.

The lessons present a series of simple activities to develop facility with the compass* and to engage the student in discovery of properties of the circle. Care and precision in drawing circles must equal that invested in lines. The compass is an ancient geometric tool with a tradition antedating the earliest works on geometry. In the beginning the compass is used only to construct circles, but its power increases with the sophistication of the student. These lessons are also an introduction to the use and the importance of the compass.

The first activities reinforce the concepts of center and radius of a circle and promote pattern recognition. Families of circles are constructed passing through a fixed point, centering on a fixed line, and centering on a fixed circle. The resulting configurations indicate relations among the various geometric elements. The second set of activities involves the construction of a perpendicular from a point to a line and of midpoints of

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* Each of these is called a "pair of compasses" because of its two legs. We will follow common usage and call each a "compass".

line segments. Symmetry about a line is implicit. The third set of activities introduce the triangle inequality through the problem of constructing a triangle with specified sides. Here the compass is also useful to "carry" or transfer lengths.

NOTE ON COMPASSES

A large board compass is necessary for class demonstrations. Each student must be supplied with a compass for the activities of this chapter. The join of the legs of a compass is its critical feature. The ideal compass has a join loose enough to admit its opening to be set with a smooth motion, but tight enough to hold the setting while a circle is being drawn. Inexpensive compasses are available, but they wear quickly. Full classroom sets of materials for CSMP Mathematics for the Intermediate Grades, Part I include 30 student compasses and one board compass. These particular student compasses can be loosened or tightened by adjusting the small screw at the top with a small screwdriver. If you already have compasses (a board compass and/or student compasses), it is possible to order your classroom materials without including the compasses mentioned above.

CAPSULE LESSON SUMMARIES

Section 2.1:	Practice with a new tool — the compass. Make simple constructions to introduce the center and radius of a circle.
Section 2.2:	By centering a family of circles on a fixed line, construct new parallel lines. By centering a family of circles on a fixed circle, construct new circles with the same center.
Section 2.3:	By centering a family of circles on a fixed line and having them all pass through a given point, construct a special second point. This is a first look at symmetry.
Section 2.4:	Find midpoints of line segments with the compass. Challenge: construct a triangle with the length of its sides specified using only a compass and a straightedge. Attacking the challenge, reinforce properties of length and the use of the compass in transferring ("carrying") lengths.
Section 2.5:	Using more triangular constructions, develop understanding of the triangle inequality and properties of length.

2.1 Circles

Distribute the compasses and give several sheets of paper to each child. Explain what a compass is for and how it works. Then give the children ten to fifteen minutes to explore the possibilities of this new tool. Most children will experience difficulty in balancing the pressure applied so that the point stays stuck in the paper and the leg that is provided with a pencil is free to move. A small amount of individual attention from you, accompanied by a demonstration and much practice will help to alleviate this difficulty. To avoid a point's tearing the drawing sheet, encourage each student to place two or three sheets under the drawing sheet. Tell your students to be sure the pencil in the compass is well sharpened at all times.

After the exploration period, draw a circle with radius 20 cm on the board. Show the children your compass.

T: I've just drawn this circle on the board with my compass. When I did that the distance from the point to the chalk was 20 cm.

Draw the following red segment using a straightedge.

T: How long is this red line segment?

A moment may pass before someone realizes that this segment must be 20 cm long.

T: Why?

S: Because when you're drawing the circle, the point of the compass is always in the middle while the chalk is drawing. If the distance from the point to the chalk is 20 cm the distance from the middle to the edge must be 20 cm.

Draw a couple of other radii of the circle, as follows:

and conclude that the above reasoning applies to all such segments and so they are all 20 cm long.

T: The place in the middle of the circle where you put the point of the compass is called the CENTER of the circle. And the distance from the center to each point of the circle, which is the same as the distance from the point of the compass to the chalk (or in your case, the pencil), is called the RADIUS[#] of the circle.

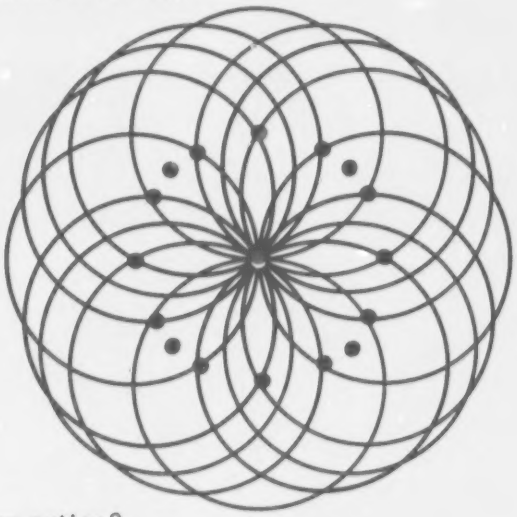
[#] The word "radius" will also be used to mean "a segment joining the center and any point of a circle". These two meanings are part of the natural language and both have already been used above in describing this lesson. Thus, the meaning of "radius" will depend on the context in which it is used.

Write these words on the board.

Make sure that everyone has a 30-centimeter ruler. Show the children how to set the opening of a compass to 3 cm by placing the point at the 0-mark and moving the pencil end until it is on the 3-mark.

T: Everybody take a clean sheet of paper and mark a small dot in blue in the middle of the page. With your compass open to 3 cm, draw a large number of circles that pass through the blue dot. Mark the center of each circle in red.

Demonstrate what it means for a circle to pass through a dot. Let the children work individually. Encourage them to check, after the drawing of about every third circle, that the openings of their compasses remain at 3 cm. Here's how a well executed drawing (reduced in size) should look: (Do not expect such well drawn, symmetric pictures from the students on their first attempt. You should draw this picture on the board after the students have worked on their own for a while.)



T: What do you notice?

S: The red dots make a circle.

T: That's right. What do you think the radius of that circle is?

This is not an easy question for the children, so be patient as you draw the answer out of them.

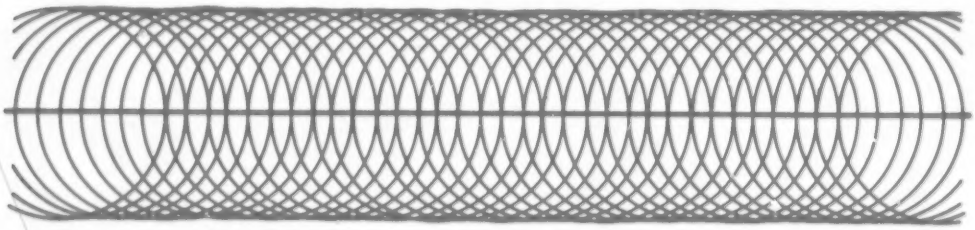
The answer is that the radius is 3 cm. Since the blue dot belongs to each of the circles and all of the circles have the same radius, the distance from the blue dot to each of the red dots is the same, namely 3 cm. To draw the complete red circle, reverse the compass. Place the point on the blue dot; the chalk or pencil will pass through the red dots.

2.2 Circles #2

Begin the lesson by reviewing the last lesson on circles, in particular, the notions of center and radius. Emphasize the fact that the radius of a circle is equal to the distance from the center to any of the points of the circle. Allow the children about 5 minutes to draw whatever they like using their compasses. Then give them the following task.

T: Use your ruler to draw a line segment (in red) lengthwise along your sheet of paper. Open your compass to 5 cm and draw a large number of circles with radius 5 cm and centers on that straight line. What do you think you will get?

Be noncommittal about the students' suggestions. Let them work on this task individually. Again, encourage them to make sure the openings of their compasses remain set at 5 cm. If anyone asks what they should do about circles that go off the page, suggest drawing just the part that is on the page. A well-executed drawing (reduced in size) will look something like the following: (After the students have worked on their own a while, draw this picture on the board making sure the students observe as you do so.)



T: What do you notice?

S: It looks like a Slinky.

S: It looks like a tunnel.

S: It's all between two straight lines.

Use a straight edge to draw in the two bounding straight lines in blue on the board.

T: Do you think either of these two straight lines will cross the red line if we kept on going long enough?

S: No. The two blue lines and the red line are all parallel.

T: How far apart are the two blue lines?

S: 10 cm.

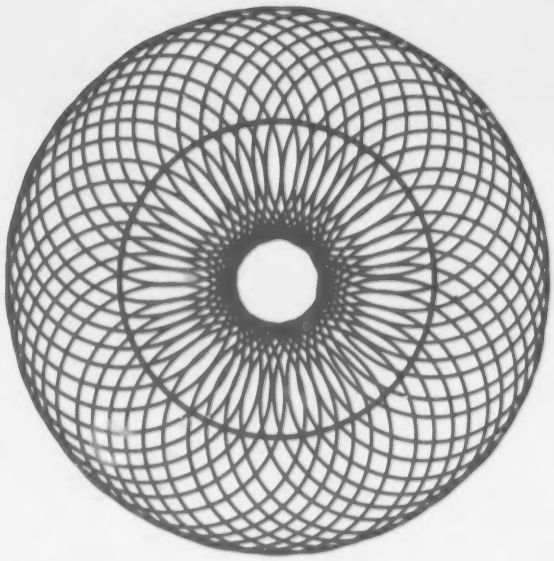
T: How do you know?

S: Well, look at a point on this blue line. It's on this circle and so it's 5 cm away from the center of the circle, which is on the red line. Then on the other side of that circle there's a point on the other blue line. It's 5 cm away from the red line too. 5 cm + 5 cm = 10 cm.

T: Very good. Take a clean sheet of paper. Here is a new problem. Draw a circle in the middle of your piece of paper with radius 5 cm. Then set your compass to 3 cm and draw a large number of circles with centers on the big circle. What do you think you'll get this time?

S: Something like a Slinky again.

Let the children work on this task as they did on the previous ones. Here is what a well-executed drawing (reduced in size) will look like: (Draw it on the board after the students have worked on their own a while.)



S: It looks like a flower.

S: It's all between two circles.

T: What's the radius of the outside circle?

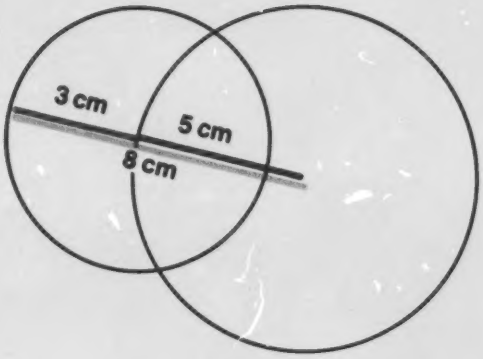
S: 8 cm.

T: How do you know?

S: The starting circle gives you 5 cm, and then the other circles take you out 3 cm more.

Help explain this reasoning to the rest of the class by drawing the following diagram on the board:

5G



Argue similarly that the radius of the inside circle is 2 cm.

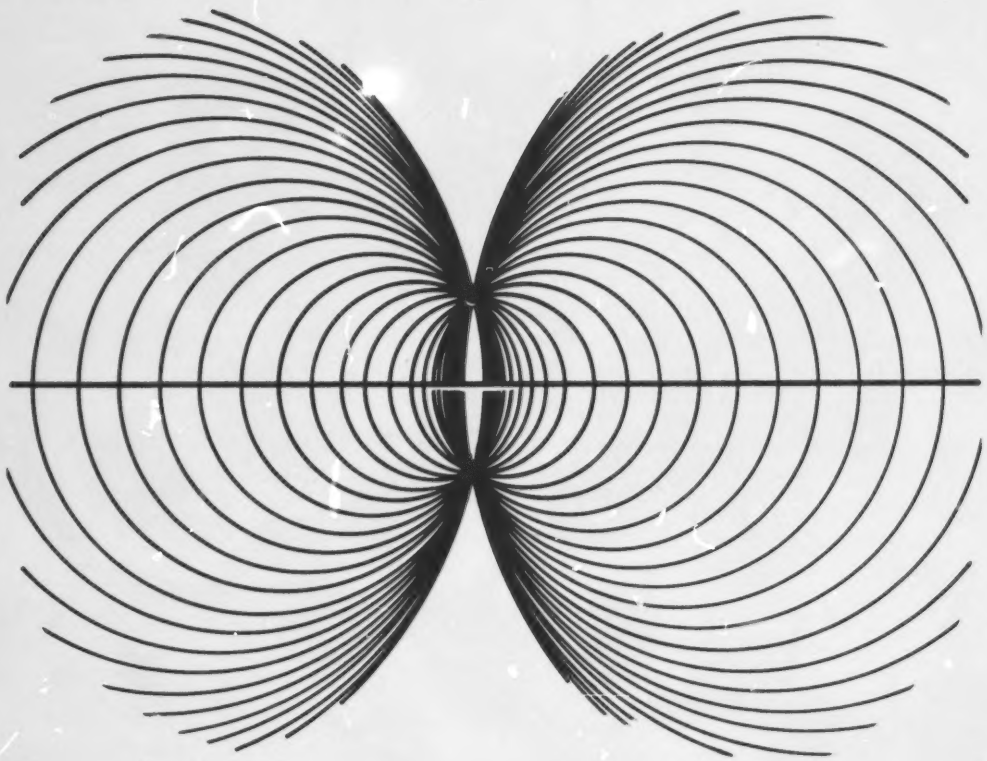
2.3 Circles #3

Proceed as in the previous lessons on circles. Present the following tasks one at a time, discussing the children's work each time.

Task 1: Draw a red line segment lengthwise along the page, and about halfway across the page mark a blue dot about 3 cm away from the line.

Draw a large number of circles whose centers are on the line and which pass through the blue dot. This time the radius of the circles will vary from one circle to the next.

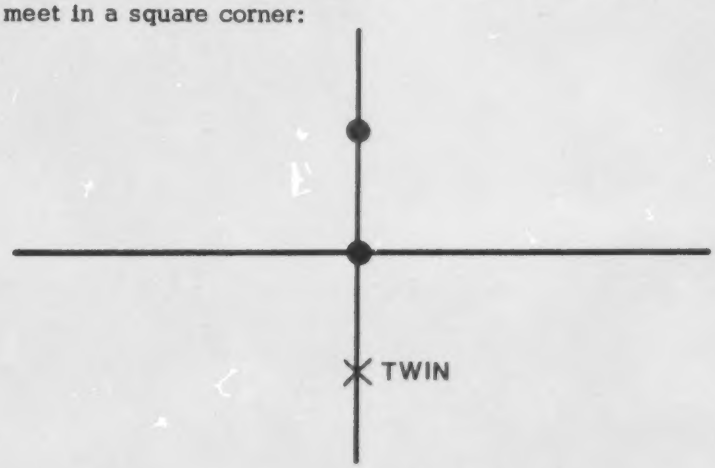
Solution:



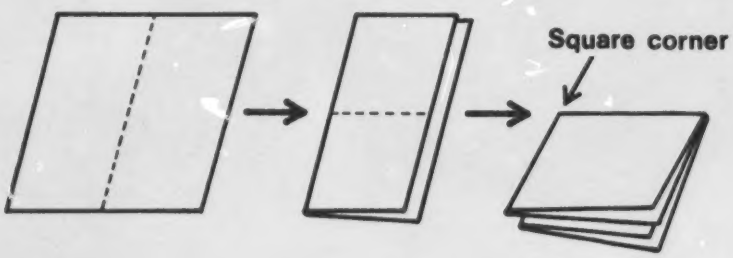
Points to emphasize:

1. There's a "twin" point on the other side of the line from the blue point. All the circles which pass through the blue point and have their centers on the red line also pass through the "twin".

2. If you draw a blue line segment passing through the blue point and its "twin", then the blue line and the red line meet in a square corner:



This can be checked using a square corner made by folding a sheet of paper:



3. The distance between the blue dot and the point where the blue and the red lines cross is the same as the distance from the twin to this crossing point. In other words, if you fold the sheet of paper along the red line, the blue dot will lie on top of its twin.

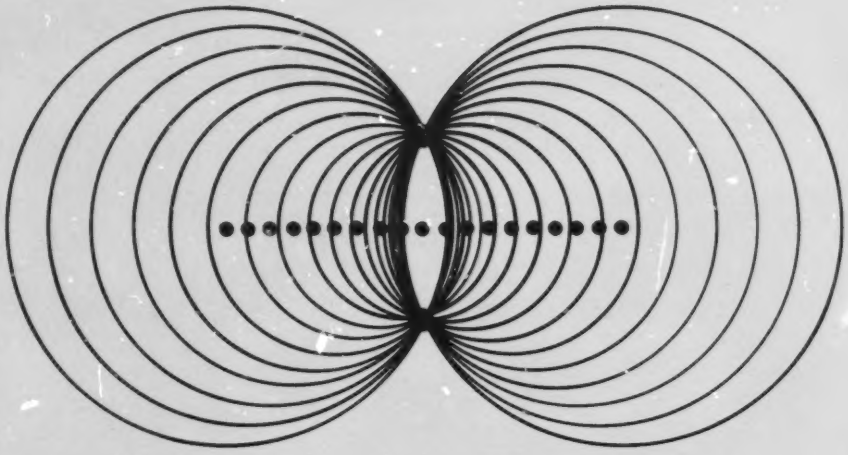
4. Restarting with the blue line and a dot marked on the red line would produce a twin for that red dot. The twin would also be on the red line on the other side of the blue line as far away from the crossing point as the original red dot.

Task 2: Draw two blue dots in the middle of the page and 7 cm apart. Draw a large number of circles which pass through both of these dots at the same time and mark the centers of the circles in red.

The mechanics of this task are not easy since it is difficult to find a place to put the point of the compass so that a circle may be drawn passing through both dots. Before letting the children work on this task individually, explain the following method of placing the point of the compass (i.e., of finding a red point):

Set your compass to any opening. Place the point at each blue dot in turn to lightly draw a circle. Any point where the circles cross can be a red dot. Notice that you do not need to draw the whole circle each time. Set your compass to a new opening of any length you want and repeat the steps above to find another red point, and so on.

Solution (reduced in size):



Points to emphasize:

1. The red dots lie on a straight line.

2. The picture is the same as that in Task 1.

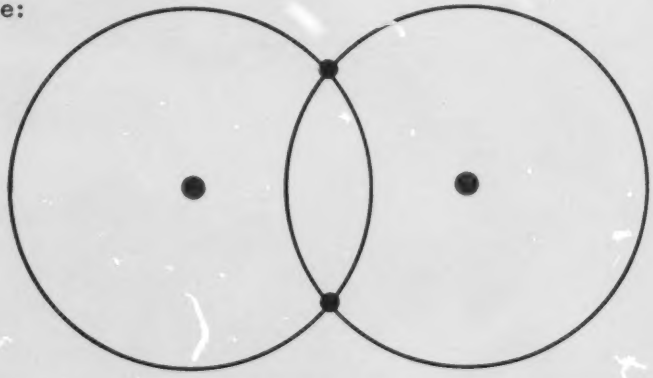
3. Everything true about Task 1 is true in this situation also.

2.4 Circles #4

Proceed as in the previous lessons on circles. Note: all of the figures in this section are reduced in size.

Task: Draw two blue dots in the middle of the page 7 cm apart. Draw pairs of circles, one with center at one blue dot and the other with the same radius and center at the other blue dot. If a pair of circles cross each other, mark the place(s) where they cross in red.

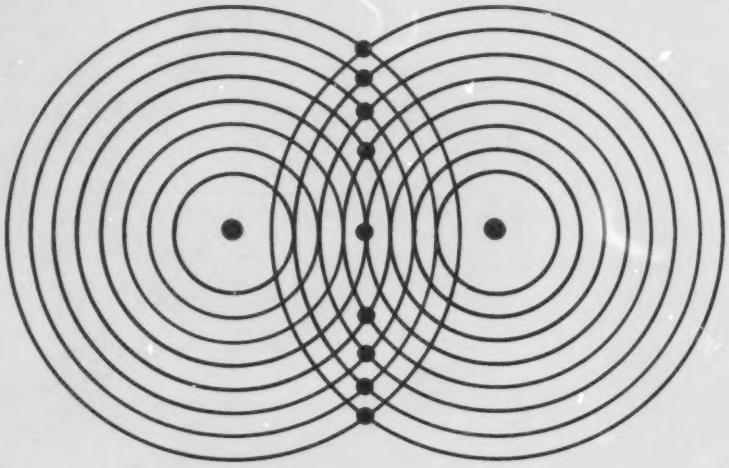
For example:



These two circles cross and we have marked their two crossing points in red.

These two circles do not cross and we mark nothing in red.

Solution:



Points to emphasize:

1. The red dots lie on a straight line.

2. The situation is similar to the two in Section 2.3.

3. If a blue line segment is drawn thrcugh the two blue dots, then the blue line and the red line meet in a square corner.

4. The red line passes through the midpoint of the segment whose endpoints are the blue dots.

5. In restarting with two of the red dots belonging to the same circle (such dots must be on opposite sides of the blue line), the pairs of crossing points would all lie on the blue line. Hence, the blue line passes through the midpoints of all the red segments whose endpoints are pairs of red points belonging to the same circle.

For the remainder of the lesson continue as follows:

Review the notion of the radius of a circle by drawing the following diagram,

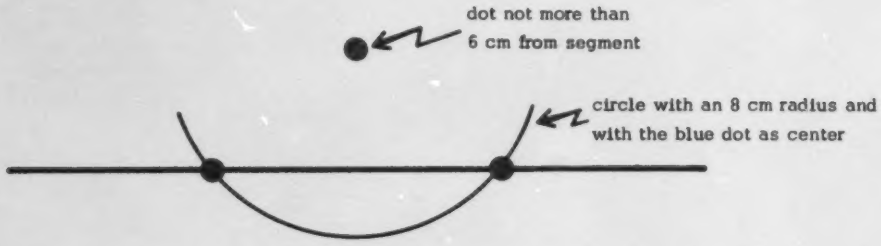
and saying:

T: The radius of this circle is 24 cm. If I mark a blue dot on the circle and I measure the distance between the red and the blue dots, what will I find?

S: The distance will be 24 cm. Every point on the circle is 24 cm away from the center.

T: Very good. Now here's a problem for you that you can solve by using what we've just been talking about. Draw a red line segment across your page and mark a blue dot not more than 6 cm away from the segment. Then use your compass to find as many red dots of the line segment as you can which are exactly 8 cm away from the blue dot.

It may take a while before anybody finds a way to do this. Here is one way:



As soon as someone finds a method, let him or her explain to the class. If the student's method is not the above method you should explain it to the class as well. Drawing a circle of radius 8 cm with center at the blue dot produces <u>all</u> points on the piece of paper which are 8 cm away from the blue dot. Thus, this circle crosses the red line at points exactly 8 cm away from the blue dot. That the circle crosses the red line is guaranteed by our picking the blue dot close enough (6 cm in this case) to the red line.

If there is time, announce the following task and let the children start working on it.

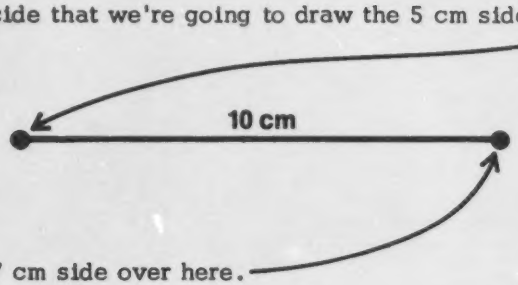
> Use your compass and your ruler to help you draw a triangle whose sides are 5 cm, 7 cm and 10 cm long.

Do not expect the students to make a transfer from the previous activity to this one. Most students will have no idea of using a compass to draw a triangle. They will discover that it's difficult to draw the required triangle <u>accurately</u> with only a ruler and pencil. It is difficult to position two of the sides so that the distance between the loose ends is equal to the desired length of the third side. If no one has an idea, after a few minutes offer a suggestion as follows:

T: Well, let's draw one side. Which side shall we draw first?

Suppose the 10 cm side is chosen.

T: Let's decide that we're going to draw the 5 cm side over here



and the 7 cm side over here.

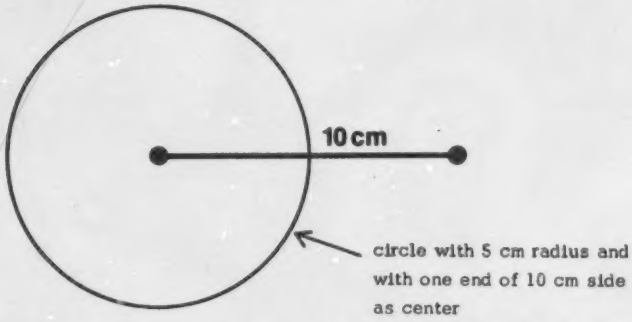
What can you tell me about the third corner of the triangle? How far away is it from the left end of the 10 cm side?

S: 5 cm.

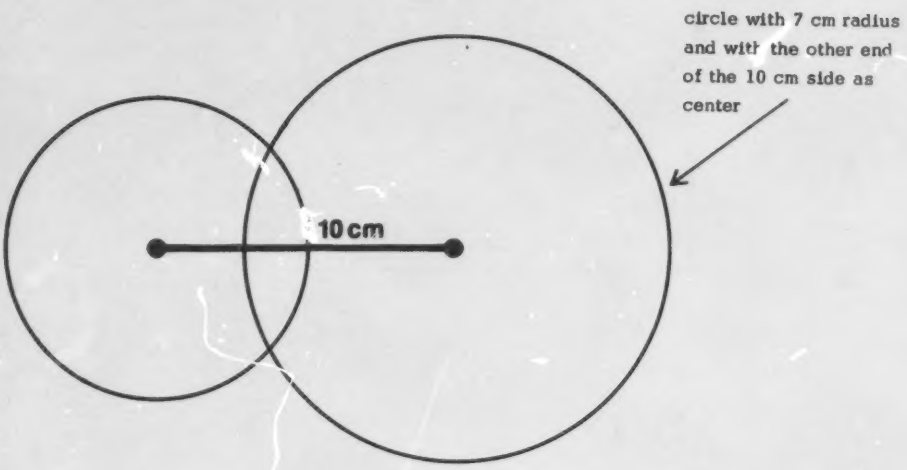
T: And how far away from the right end?

S: 7 cm.

S: I know! You draw a circle. Like this:



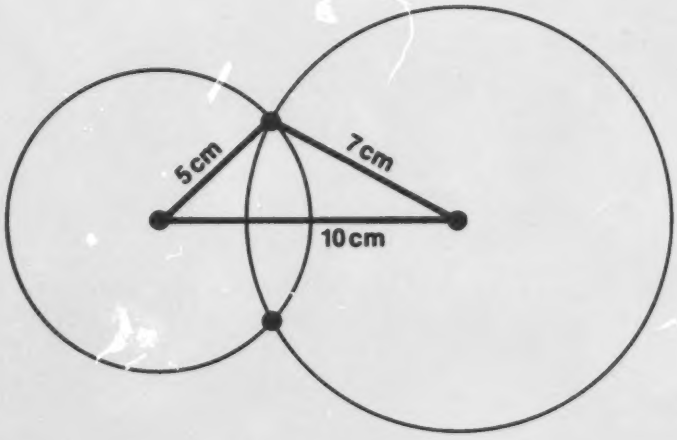
S: Then you draw another circle on the other end. Like this:



S: Where they cross is the third corner.

S: There are two possibilities.

T: You can choose either one. I'll complete the triangle with the top crossing point.



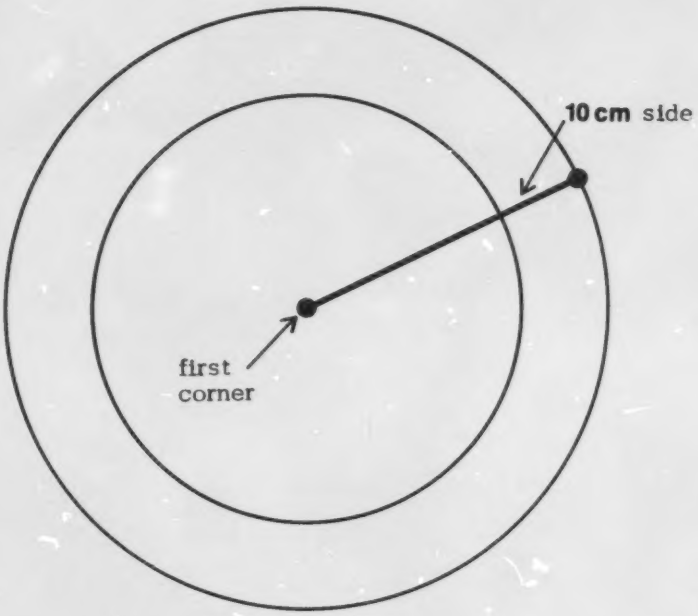
Do not force the children through the above method if someone invents another one. Sometimes the methods which the children generate are ingenious and surprising. For example, one fourth grade girl solved the problem as follows:

1. Draw one corner of the triangle.

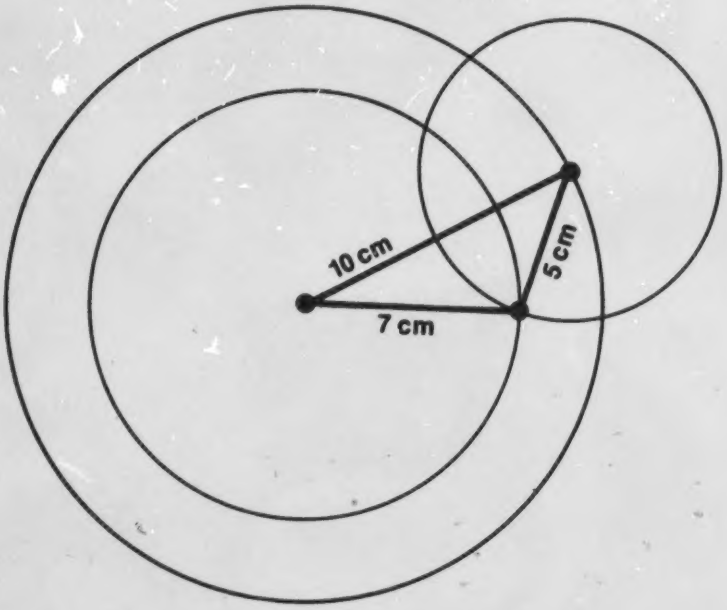
2. Draw a circle whose center is that corner and whose radius is 10 cm. The other end of the 10 cm side of the triangle is somewhere on that circle.

3. Draw a circle whose center is the first corner and whose radius is 7 cm. The other end of the 7 cm side is somewhere on that circle.

4. Draw the 10 cm side at random.



5. Draw a circle of radius 5 cm with center at the outside end of the 10 cm side. Where it crosses the 7 cm-radius circle is the third corner (two possibilities - only one drawn here):



2.5 The Triangle Inequality

If you have not already done so, go through the activity described at the end of Section 2.4. However, if you do this, precede it with a couple of brief warm-up exercises such as those used just before the last activity of Section 2.4.

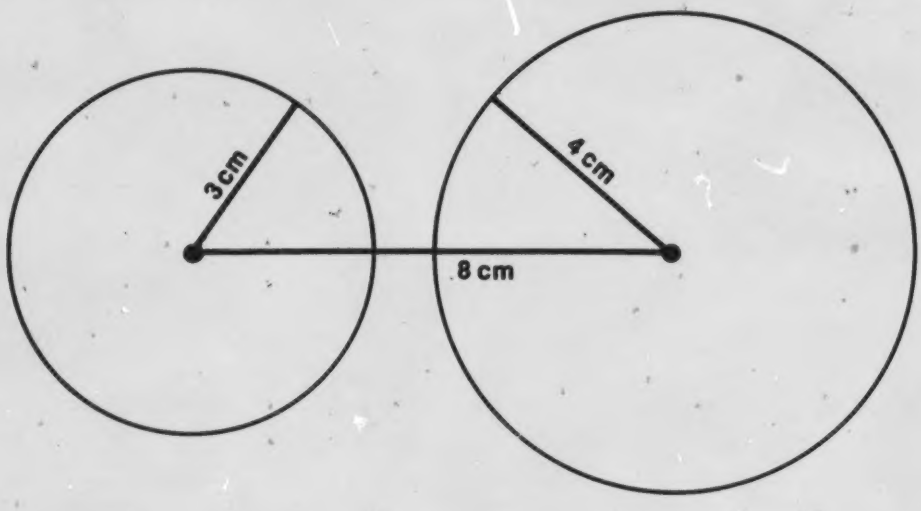
Then proceed to the following similar activities:

1. Draw a triangle whose sides are 6 cm, 8 cm and 10 cm long.

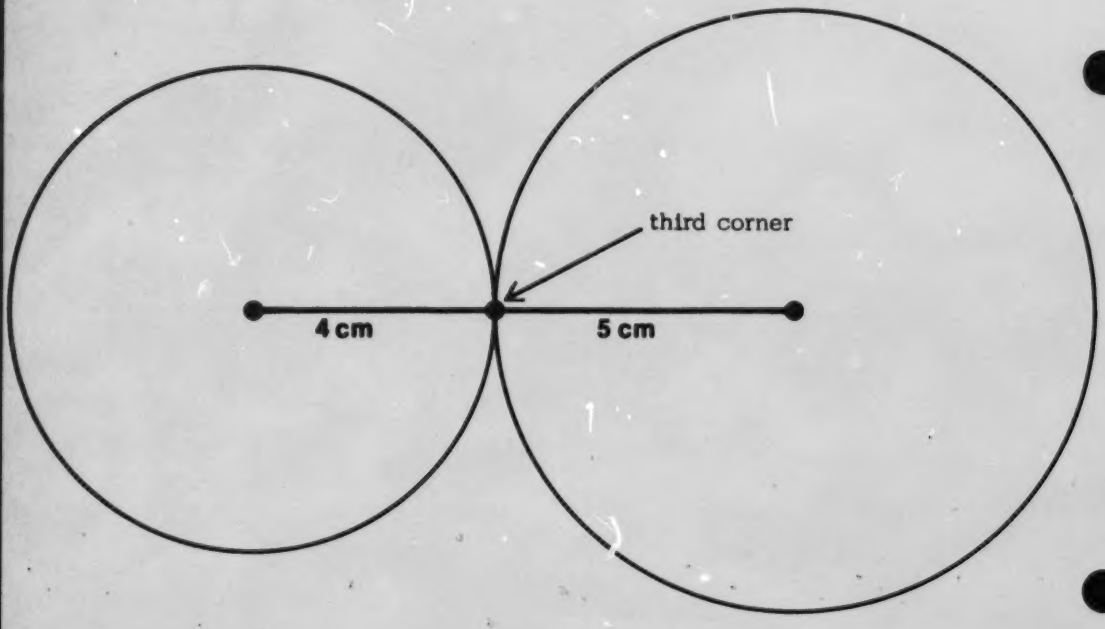
2. Draw a triangle whose sides are 3 cm, 4 cm and 8 cm long.

3. Draw a triangle whose sides are 4 cm, 5 cm and 9 cm long.

The children will discover that the second of these is impossible since the two circles will not meet.



And the third doesn't yield a triangle because the third corner is <u>on</u> the 9 cm side.



T: It seems that with some of these problems you can make a triangle and with others you can't. How can you decide if you can make a triangle?

S: If for <u>every</u> two sides their total length is more than the length of the other side, then you can make a triangle.

T: What about if the total length of two of the sides is exactly the same as the length of the other side?

S: Then you end up like that last problem where the third corner was on the first side. So no triangle is possible in this case.

T: And what if there are two sides whose total length is less than the length of the other side?

S: Then you can't make a triangle.

Give the children the following problems. In each case, they are to decide if one can construct a triangle out of sides of those lengths and, if so, to make such a triangle.

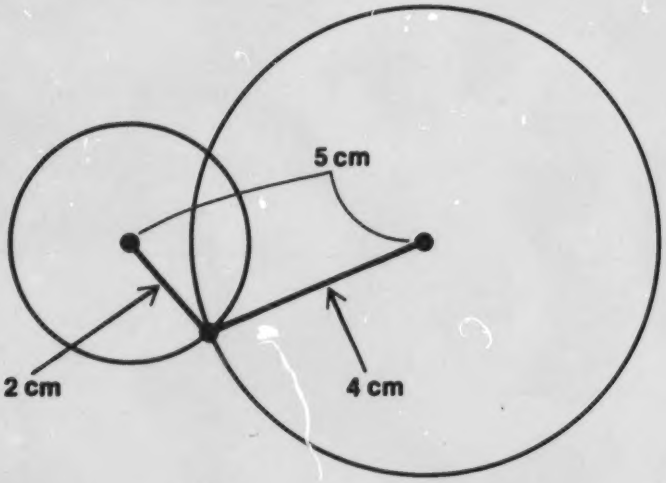
1. 11 cm, 6 cm, 5 cm. (No, since $6 + 5 = 11$)
2. 3 cm, 12 cm, 8 cm. (No, since $3 + 8 < 12$)
3. 5 cm, 12 cm, 8 cm. (Yes, since $5 + 8 > 12$, $5 + 12 > 8$ and $12 + 8 > 5$)

If time and interest permit, investigate some of the following problems, which are more of the first type in disguise.

1. Draw two dots 5 cm apart. Draw a two-part zigzag with a total length of 6 cm going from one dot to the other.
2. Draw two dots 9 cm apart. Draw a two-part zigzag with a total length of 11 cm going from one dot to the other.
3. Draw two dots 6 cm apart. Draw a two-part zigzag with a total length of 5 cm going from one dot to the other. (Of course, there is no solution to this problem.)

Problems such as these can be generated at will and you can progress to 3-part, 4-part, 5-part, etc. zigzags if your students are interested.

The difference between the zigzag constructions and the triangle constructions lies in the freedom of deciding where the bend in the zigzag will be. For example, in the first case, deciding to bend the zigzag 4 cm from one end results in the following:



Probability and Statistics

Comprehensive School Mathematics Program

Mathematics for the Intermediate Grades Part I

Probability and Statistics

CEMREL, Inc.

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SUGGESTED SCHEDULE FOR TUESDAY'S AND THURSDAY'S LESSONS

WEEK	TUESDAYS	MATERIALS
1	1L L1.1: The String Game with A-Blocks p. 2	Teacher: Colored chalk; The String Game; 2 calculators (optional)* Student: None
2	2L L1.2: The String Game with Numbers: Multiples and Order p. 12	T: Colored chalk; The String Game S: None
3	3L L2.1: Logical Thinking #1 p. 52	T: Colored chalk; The String Game S: Worksheets L01, L02; pencils
4	4L L2.2: Logical Thinking #2 p. 61	T: Colored chalk; The String Game S: Pencils; paper; Worksheets L03, L04
5	4G G2.1: Circles #1 p. 39	T: Compass; straightedge; colored chalk S: Compass; 30 cm ruler; colored pencils
6	5G G2.2: Circles #2 p. 43	T: Compass; straightedge; colored chalk S: Compass; 30 cm ruler; colored pencils
7	6G G2.3: Circles #3 p. 47	T: Compass; straightedge; colored chalk S: Compass; 30 cm ruler; colored pencils
8	FREE DAY	
9	7G G1.4 Area Problems p. 29	T: Grid board; Flip flop aid; colored chalk S: Worksheets G06, G07; colored pencils

SUGGESTED SCHEDULE FOR TUESDAY'S AND THURSDAY'S LESSONS

THURSDAYS	MATERIALS
1G G1.1: Comparisons of Lengths and Areas p. 3	T: Grid board; colored chalk S: 30 cm rulers; Worksheets G01, G02; colored pencils
2G G1.2: Introduction to Millimeters p. 12	T: None S: 30 cm rulers; Worksheets G03
3G G1.3: Square Regions p. 19	T: Four colored squares; grid board; colored chalk S: Colored pencils; Worksheets G04, G05
FREE DAY	
1P P1.1: Same or Different #1 p. 2	T: Marbles; colored chalk S: Marbles; pencils; paper
2P P1.2: Same or Different #2 p. 11	T: Marbles; colored chalk S: Pencils; paper
3P P1.3: Same or Different #3 p. 16	T: Colored chalk S: Pencils; Worksheets P01, P02, P03, P04, P05, P06, P07
4P P1.4: Same or Different #4 p. 22	T: Marbles; colored chalk S: Marbles; colored pencils; pencils; paper
5P P1.5: Same or Different #5 p. 30	T: Marbles; colored chalk S: Marbles; colored pencils; pencils; paper

SUGGESTED SCHEDULE FOR TUESDAY'S AND THURSDAY'S LESSONS

WEEK	TUESDAYS	MATERIALS
10	5L p. 80 L3.1: Composition Games #1	T: Colored chalk S: Colored pencils; Worksheets L09, L10, L11
11	6L p. 90 L3.2: Composition Games #2	T: Colored chalk S: Colored pencils; Worksheets L12, L13, L14
12	8L p. 99 L3.3: Composition Games #3	T: Colored chalk S: Colored pencils; paper
13	10L p. 105 L3.4: Composition Games #4	T: Colored chalk S: Colored pencils; paper; pencils
14	FREE DAY	
15	FREE DAY	
16	6P p. 36 P2.1: "The Square Trap" #1	T: "The Square Trap"; Minicomputer Set S: "The Square Trap"; Worksheet P08; spinner arrows; Minicomputer checkers; pencils
17	7P p. 38 P2.2: "The Square Trap" #2	T: "The Square Trap"; Minicomputer Set S: "The Square Trap"; Worksheet P08; spinner arrows; pencils; paper
18	8P p. 43 P2.3: "The Square Trap" #3	T: "The Square Trap"; Minicomputer Set S: "The Square Trap"; Worksheet P08; spinner arrows; pencils; paper

SUGGESTED SCHEDULE FOR TUESDAY'S AND THURSDAY'S LESSONS

THURSDAYS	MATERIALS
FREE DAY	
7L p. 67 — L2.3: Logical Thinking #3	T: Colored chalk; The String Game S: None
9L p. 72 — L2.4: Logical Thinking #4	T: Colored chalk; The String Game S: Worksheets L05, L06, L07, L08; pencils
FREE DAY	
8G p. 51 — G2.4: Circles #4	T: Compass; straightedge; colored chalk S: Compass; 30 cm rulers; colored pencils
9G p. 59 — G2.5: The Triangle Inequality	T: Compass; straightedge; colored chalk S: Compass; 30 cm rulers; colored pencils
11L p. 14 — L1.3: The String Game with Numbers: Multiples, Divisors and Order #1	T: Colored chalk; The String Game S: Pencils; paper; colored pencils
12L p. 21 — L1.4: The String Game with Numbers: Multiples, Divisors and Order #2	T: Colored chalk; The String Game S: Colored pencils; pencils; paper
13L p. 27 — L1.5: The String Game with Numbers: Multiples, Divisors and Order #3	T: Colored chalk; The String Game S: Colored pencils; pencils; paper

SUGGESTED CHRONOLOGICAL ORDER OF "P" LESSONS

Lesson One:	P1.1 - Same or Different #1	2
Lesson Two:	P1.2 - Same or Different #2	11
Lesson Three:	P1.3 - Same or Different #3	16
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PROBABILITY AND STATISTICS

INTRODUCTION

There are several reasons why people study mathematics. It is an interesting and challenging intellectual exercise when studied for its own sake. But it is also a supremely useful topic of study in that it has applications outside itself—to other disciplines, as well as more generally to understanding the world around us. Some areas of mathematics are more easily applied than others, and the areas which particularly concern us in this strand are towards the easy end of the spectrum, both because of the small amount of background mathematical knowledge required before meaningful application can be made, and because it is usually fairly easy to see the link between the mathematical model and the "real" situation being modeled. For these reasons, it would be difficult to overstate the importance in today's world of probability and statistics and their applications to problems and situations within and outside the discipline of mathematics.

The lessons of this strand deal solely with the exploration of two situations involving probability. (Part II of the CSMP Intermediate Grades Program will include a Statistics component.) These two situations extend types of activities similar to those presented in Part II of the CSMP Upper Primary Grades Program. In addition, however, systematic methods of analyzing such situations are introduced for the first time.

The first situation (Chapter 1) develops as a series of games of a relatively simple nature, called "Same or Different". In each lesson, along with the experimental activity of collecting frequency data, the language of arrows (and cords) is used to analyze the various games and explicitly determine numerical probabilities for the possible outcomes.

The situation in Chapter 2 is the series of games presented in the storybook THE SQUARE TRAP, one of the collection, STORIES BY FREDERIQUE. These games are more complex than those of Chapter 1. The activity remains at the experimental level. The main object is the exploration of the idea of expectation which arises through a process of averaging the outcomes of a large number of games. One striking result, arrived at through experimentation, is that two seemingly quite different games are actually equivalent. We refer you to THE SQUARE TRAP for further details and explanations.

1

SAME OR DIFFERENT

This series of lessons deals with a collection of games that are played with marbles. In the initial situation, one has a certain number of marbles of two colors, and from this set two marbles are chosen at random. The results are recorded according as the two marbles are of the same or of different colors. The game is then generalized in some of the many possible ways, and each generalized situation is analyzed.

In the initial stages, each game is actually played by pairs of children and the results thus obtained are discussed. Then each game is analyzed mathematically and the probabilities of the various outcomes are derived. Predictions are confronted with actual data. Later on, experimentation is abandoned and attention is focused on the mathematical analysis, the derivation of probabilities and the prediction of expected frequencies.

CAPSULE LESSON SUMMARIES

Section 1.1: Using a number of marbles of two colors, choose two marbles at random. After repeated trials obtain the frequencies of the same or different colors, and determine the probabilities by analyzing the situation.

Section 1.2: Extend the games of Section 1.1 to consider a greater number of marbles. Discuss strategies for predicting individual outcomes.

Section 1.3: Extend the material in Sections 1.1 and 1.2. (Worksheets P01, P02, P03, P04, P05, P06, P07)

Section 1.4: Modify the rules for the "Same or Different" games. Play the games by choosing a marble, replacing it, and choosing again.

Section 1.5: Examine situations in which two different games have the same probabilities for "different" and "same". Also, analyze a game with three outcomes.

1.1 Same or Different #1

Take one red and two blue marbles and tell the children what you have. Explain that you are going to play a game, and then show them how it is played.

1. Mix up the marbles by shaking them together in your cupped hands.
2. Without looking, transfer the marbles to one hand and put your hands behind your back.
3. Take two marbles in one hand and show the class which two you have. (Be sure to keep your other hand containing the remaining marble behind your back.)

Alternatively, at Step 3 you could let the students predict whether the two marbles will be of the same or of different colors before you show them. In any case, explain that what you are interested in is whether the two marbles are

of the same color (recorded "S")

or

of different colors (recorded "D").

Ask for two volunteers to help you play the game. One is to shake and randomly choose the two marbles, and the other is to announce whether the chosen two are of the same or of different colors. Before they begin, you could ask the class to predict how many "S"'s and how many "D"'s they expect to result from playing the game ten times. Each prediction should be accompanied by an explanation, which should be accepted noncommittally. The two students should play the game ten times as you record the results on the board. Here is what might happen:

D: 𝍸 |

S: ||||

Discuss these results briefly, but point out that ten trials do not provide enough information to allow you to draw any conclusions about the game.

T: So I'm going to divide you up into pairs, and each pair will play ten games. How many pairs will we have?

If you have an odd number of children, explain that one child will play ten games alone.

T: How many times will we play the game altogether?

If you have 30 students in your class, you are looking for an answer such as:

S: We'll have 15 pairs; each pair plays ten times. That makes $15 \times 10 = 150$ games.

T: In about how many of those 150 games do you think we will end up with marbles of different colors? of the same color?

Accept whatever responses are offered by your children. Allow them to explain their thinking if they want to, but don't insist on explanations at this stage. You may wish to write the predictions on the board and to compare them later with the actual data you obtain.

Divide your class into pairs, and assign one of each pair to deal with the marbles and the other to record the results. Distribute the necessary marbles to the children, and, before letting them start playing their ten games, explain that it is most important to give the marbles a good shake each time because otherwise strange things can happen.

Let the children start playing their ten games. As the pairs finish their ten games they should tell you their results, which you should then record on the board. In a typical class of 30 students, the results might be as follows:

D	S
7	3
8	2
8	2
7	3
7	3
7	3
8	2
3	7
9	1
6	4
6	4
7	3
5	5
8	2
7	3

T: What do you notice about these results?

S: We got "D" more times than we got "S".

Total the "D-column" as a class, using the "running total" method:

$$7 + 8 = 15;\ 15 + 8 = 23;\ \ldots;\ 96 + 7 = 103.$$

T: Can we find the total of the "S-column" without actually adding all those numbers?

S: We played the game 150 times altogether. 103 of them resulted in "D", so 47 of them must have ended in "S", since 150 - 103 = 47 (or 103 + 47 = 150).

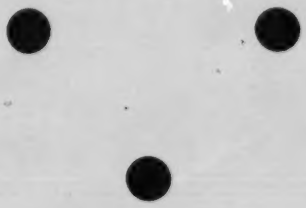
To check your work, total the "S-column" in the same way as you totaled the "D-column".

T: Do you think you are more likely to get "D" or "S" in this game?

S: It looks as if you're more likely to get "D".

T: Let's find out if that's really so; and if it is, why it is. Here are the three marbles: two blues and a red.

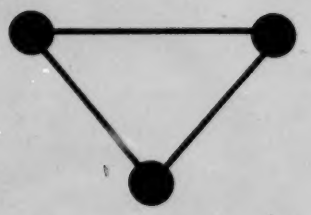
Draw on the board:



T: Here is one way to take two marbles.

Draw the following stroke linking the red and one of the blue marbles:

Ask a volunteer to come to the board and indicate the other possible ways of taking two marbles.



T: So altogether there are three possibilities. In how many of them do you get a pair of marbles of the same color?

S: One.

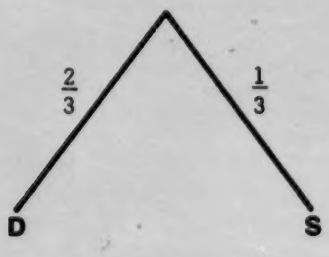
Indicate the stroke linking the two blue marbles.

T: So we have one chance out of three of getting "S". How many chances out of three do we have of getting "D"?

Indicate the other two strokes.

T: Here's another way of showing what we have found out about this game.

Draw the following on the board, explaining as you do so that "$\frac{2}{3}$" means "two chances out of three", and "$\frac{1}{3}$" means "one chance out of three".



T: Do you see what this means? It means that every time you play this game you are more likely to get "D" than you are to get "S". That's because two chances out of three is better than only one chance out of three. Something else it means is that if you play the game several times you can expect that about two-thirds of the time you'll get "D" and one-third of the time you'll get "S". So what could we have expected to get in the 150 games we just played? About how many times could we have expected to get "S"? Why?

S: About 50 times, because $\frac{1}{3} \times 150 = 50$ (or $150 \div 3 = 50$).

T: About how many times could we have expected to get "D" in our 150 games? Why?

S: About 100 times, because $\frac{1}{3} \times 150 = 50$ and $2 \times 50 = 100$ (or because $150 - 50 = 100$).

T: How does that compare with what actually happened?

S: Pretty well. 100 to 50 is pretty close to 103 to 47.

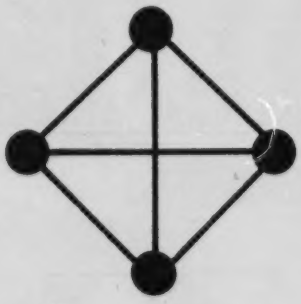
T: So you see, if we had known about this picture (indicate the "probability tree") before the game started we wouldn't have had to guess so wildly about what might happen. You'll find out how this works out in just a minute, because now we're going to play a different game. I'm going to add one more blue marble. That will make one red marble and three blue marbles. We'll play the game the same way: by taking two marbles without looking and then seeing whether they are of the same or of different colors. How do you think the results will change, if at all? Do you think we'll get 103 to 47 again? Do you think it will be more or less likely to get "S" in this game than it was in the first game? About how many times do you think

we'll get "S" if we play this new game 150 times just as we did the first game?

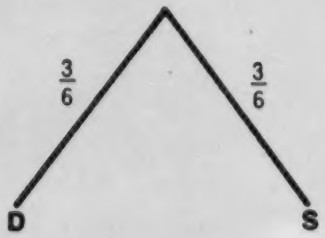
S: There are more blue marbles this time, so you're more likely to get two blue marbles.

T: Before we actually play the game, let's find out what the chances for "S" and for "D" really are.

Draw a picture of the four marbles on the board and let the students find the six possible ways of taking two marbles. You will then have the following picture:



As in the case of the first game, discuss with the students how this yields the following picture of the game:



T: Now we will play this game 150 times, just as we did the first game. How many times do you expect we will get "D"? How many times do you expect we will get "S"?

S: There will be about 75 "D"'s and 75 "S"'s because they're equally likely.

Once again, before letting the children play the game, remind them how important it is to give the marbles a good shake so as to mix them up well before each draw. Indeed, if you notice that several pairs of children are turning in markedly skewed results, it is probably due to their not mixing the marbles sufficiently well between draws. In addition, it is especially important when the children play this game that they keep the hand they are not showing behind their back. This is because, since this game uses four marbles, they will have two marbles in each hand, and hence might forget which one they are supposed to be showing. In certain circumstances this could bias the results. Let the students play the game in pairs as before. Record each pair's final results as they are obtained. When all the scores are in, total the two columns as before and discuss whether or not the results were as expected. In practically all cases you should have a fairly even split between "D" and "S".

Point out to the children what a difference it makes being able to figure out in advance what to expect. Contrast the situation in the first game, where no one had any real knowledge beforehand about what might happen—you just had to play and see what happened—with the situation in the second game, where, by doing a little hard thinking, you were able to discover about how many times "D" and "S" would occur before you even laid hands on the marbles. Then the data obtained by actually playing was merely a verification of the thinking you had done previously.

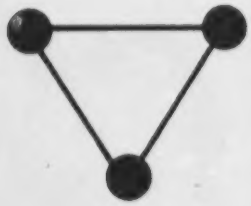
Finish the lesson by pointing out the following interesting fact: When you are playing this particular game, just before you look to see whether the two marbles you have chosen have the same or different colors, you have two marbles in each hand. In one hand you must have two blue marbles, and in the other hand you must have one red and one blue marble. There is one chance out of two that the hand whose contents you are about to inspect will be the one containing two marbles of the same color; similarly there is one chance out of two that it will be the hand containing two marbles of different colors. Hence, the game could equally well be represented by this picture:

This means, of course, that $\frac{3}{6} = \frac{1}{2}$.

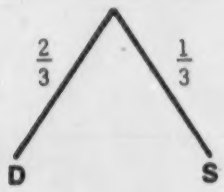
As reinforcement for this idea, you could also point out that $3 = \frac{1}{2} \times 6$ and $1 = \frac{1}{2} \times 2$, or you could show a cake cut into six equal pieces with three of them distinguished in some way:

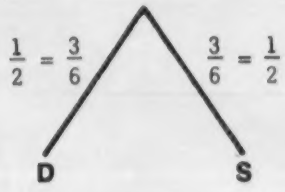
1.2 Same or Different #2

Start this lesson by giving a brief review of the lesson of Section 1.1. Draw the pictures of the marbles involved in the two games played during that lesson, and show all the different ways of taking two marbles.



Remind the children that another way of picturing these two games is as follows:





where "D" and "S" stand for "different colors" and "same color", respectively. At this stage you should inform the children that the numbers which indicate how likely it is to get "D" and "S" in each game are called the probabilities of those outcomes. Thus, for example, in the first game, the probability of getting the outcome "D" is $\frac{2}{3}$, and, in the second game, the probability of getting "D" is $\frac{3}{6}\left(=\frac{1}{2}\right)$. In the second game, the two outcomes, "D" and "S", are equally likely: they have the same probability, $\frac{1}{2}$.

T: Suppose we played the second game 60 times. About how many times do you expect we would choose two marbles of different colors?

S: That happens about one-half of the time. So I would expect to get "D" about 30 times, because $\frac{1}{2} \times 60 = 30$ (or $60 \div 2 = 30$).

T: And about how many times do you expect to get "S"?

S: About the same number of times, about 30 times.

T: Suppose we played the <u>first</u> game 60 times. About how many times would you expect to choose two marbles of the same color?

S: That happens about one-third of the time. So it should happen about 20 times, because $\frac{1}{3} \times 60 = 20$ (or $60 \div 3 = 20$).

T: And how many times do you expect to get "D"?

S: About 40 times, because $60 - 20 = 40$ (or because $\frac{1}{3} \times 60 = 20$ means that $\frac{2}{3} \times 60 = 2 \times 20 = 40$).

T: Do you think we would choose two marbles of different colors <u>exactly</u> 40 times?

S: Probably not, but it will be somewhere around 40 times, say 37, 38, 39, 40, 41, 42, or 43 times.

Conduct a similar discussion of what the children expect to happen when each of the games is played, say 120, 150, and 240 times. In addition, discuss what might be expected to happen if the games are played 100 times each. Notice that in the case of the first game, the theoretical result:

D: $66\frac{2}{3}$

S: $33\frac{1}{3}$

has no practical significance. Hence it makes much more sense in this case

to say that we expect "D" to occur about 67 times, and "S" about 33 times.

T: Suppose now that I was playing the game with one red and two blue marbles. As usual, I'm going to choose two marbles without looking and you have to say whether the two marbles will have the same or different colors. BUT, I'm only going to do it once: you get just the one opportunity to make your prediction. However, if you are right you would get a large, valuable and desirable prize! What would you predict, "D" or "S"?

S: "D", because "D" has two chances out of three; "S" only has one chance out of three. It's more likely that the colors will be different than that they will be the same.

This is a most important point: in a situation such as this, the best you can do is predict that the most likely outcome will in fact occur.

Next, go on to discuss this "one time" prediction situation in the case of the second game. Here the children should realize that it doesn't make any difference whether you predict "D" or "S". Since the two outcomes are equally likely, you're just as likely to be right as wrong, whichever outcome you predict.

Now, divide the class into two teams, of as close to equal size as possible and let these teams play the following game. One team is called the "Shakers" and the other is called the "Forecasters". You take in your hand one red marble and three blue marbles. Approach the first member of the Shakers and give him or her the red marble, then explain that you may be asked either for two or for all three of the blue marbles. Make sure everyone, and in particular the Forecasters, know what the Shaker asked for.

Having obtained from you the blue marbles he or she requested, the first Shaker goes through the routine of randomly choosing two marbles and holding out his or her clenched fist containing the chosen pair. At this point the first Forecaster has to forecast whether the chosen two marbles are of the same or of different colors. The Shaker reveals the contents of his or her clenched fist. If the Forecaster made a correct forecast, the Forecasters are given one point; if the forecast was incorrect, the Shakers are given one point. Take back all the marbles, and proceed with the second members of the teams, and so on. Continue until each member of each team has had a turn (if you have an odd number of students, the first member of one team will have to have a second turn). Finally, total the teams' scores and see which team has won.

Next, switch the roles of the teams and play the game again.

After you have finished playing the game, discuss what strategies seem to be appropriate for each team. If the children raise these questions themselves while the play is still in progress, then go ahead and discuss them at that time. (This may mean that, after a certain time, there is no point in playing any further.) The points to be brought out are the following:

(i) The Shakers should make it as hard as they can for the Forecasters to forecast correctly.

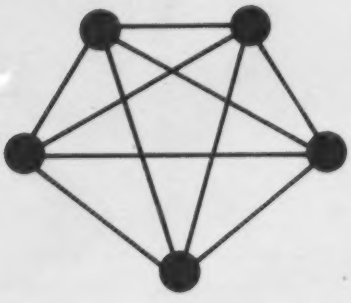
(ii) If the Shakers choose two blue marbles, then the Forecasters know that "D" has probability $\frac{2}{3}$, and so they should forecast that the colors will be different.

(iii) Hence the Shakers should always choose three marbles, and then the Forecasters are faced with a choice between two equally likely outcomes.

In other words, the game is at best fair (if (iii) above is followed); in all other circumstances the Forecasters are favored.

If time allows, you could finish the lesson by asking the children to think about the "Same or Different" game which involves one red and four biue marbles. Draw the five marbles on the board; ask the children to copy this drawing onto their pieces of paper, show all the ways of choosing two marbles, and find the probabilities for "D" and "S" in this case.

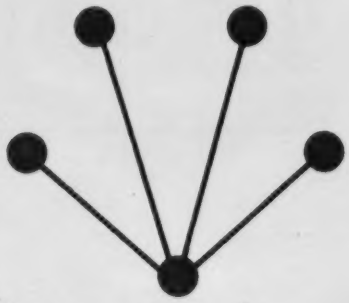
The solution is shown below:



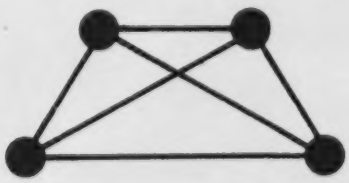
1.3 Same or Different #3

By way of review, go through the analysis of the "Same or Different" game for one red and four blue marbles, even if you already did this at the end of the lesson of Section 1.2. Here is the way we suggest you proceed:

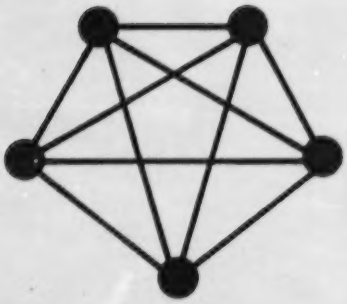
1) Draw a picture of the one red and four blue marbles and ask a student to indicate all the possible ways to choose two marbles of different colors.



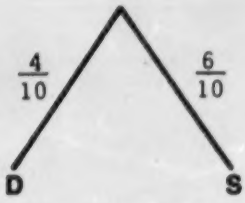
2) Beside this picture, draw the one red and four blue marbles again. This time ask a student to indicate all the possible ways to choose two marbles of the same color.



3) Finally, draw a third copy of the one red and four blue marbles, next to the previous two, and ask a student to indicate all the possible ways to choose two marbles.



4) Ask the children about the probabilities of the two outcomes "D" and "S". Record the information on the board as follows:



T: Which has the larger probability: "D" or "S"?

S: "S"; it has six chances out of ten, which is more than four chances out of ten.

T: So, when you choose two marbles, are you more likely to get different colors or the same color?

S: Same color; the probability for that is more than the probability of getting different colors.

T: Are the chances very good that you will get two marbles of different colors?

S: They're not too bad. $\frac{4}{10}$ is not a lot less than $\frac{6}{10}$.

T: If you played the game 100 times, about what would happen?

S: You would get "D" about 40 times and "S" about 60 times.

T: Suppose I were to play this game once only and I asked you to predict whether I would get different colors or the same color. If you predicted correctly, I would give you a valuable prize. What would you predict?

S: Same color.

T: Why?

S: Because "S" is more likely to happen than "D". That means I would be more likely to be right if I said you would get the same color.

T: If I asked you to predict what would happen each game for 200 games, and you predicted the same color each time, about how many times would you be right?

S: About 120 times. And I would be wrong about 80 times.

T: How did you figure that out?

S: "S" is going to happen about $\frac{6}{10}$ of the time. $\frac{1}{10} \times 200 = 20$, so $\frac{6}{10} \times 200 = 6 \times 20 = 120$.

Distribute Worksheet P01. Explain that this two-sided worksheet simply repeats the situation at the beginning of this lesson, and that the students will find it useful to keep this example for reference while they are working on the other worksheets that go with this lesson.

Then distribute Worksheet P02 and tell the children to begin work. As students successfully complete each worksheet they should be given the next (P03, P04, P05, P06, P07). You will notice that, to begin with, the children have a marked tendency to rush through each worksheet. You should discourage this as follows:

1) Keep the answer keys to the worksheets handy.

2) As a child claims to have finished a worksheet, skim through his or her work checking it against your answer key and looking for the first error.

3) If there is no error, give the child the next worksheet.

4) If you find an error, tell the child that he or she has made a mistake, point out where it is, and tell him or her to check that part and all subsequent parts of the worksheet.

If a child successfully completes all of the worksheets before the end of the lesson, suggest that he or she think up some similar problems and then solve them, for example, with six blue and two red marbles.

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Name _____ P02 *

Here are two red and two blue marbles.

In the following picture show all possible ways of taking two marbles of the same color (both blue or both red).

How many ways? 2

In the following picture show all possible ways of taking two marbles of different colors.

How many ways? 4

Turn the page

In the following picture show all possible ways of taking two marbles.

How many ways? 6

Write the probabilities next to the two lines in this picture: *

Are you more likely to get different colors or the same color? different colors

Which has the larger probability, D or S? D

* Since $\frac{4}{6} = \frac{2}{3}$ and $\frac{2}{6} = \frac{1}{3}$, $\frac{2}{3}$ and $\frac{1}{3}$ are also correct answers for the probabilities of D and S respectively. So too, of course, are any other fractional names for $\frac{2}{3}$ and $\frac{1}{3}$.

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Name _____ P03 *

. Here are two red and three blue marbles.

In the following picture show

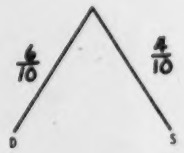
all possible ways of taking two marbles of different colors.

all possible ways of taking two marbles of the same color.

all possible ways of taking two marbles.

Turn the page

Write the probabilities next to the two lines in this picture: *



Are you more likely to get different colors or the same color? different colors

* Since $\frac{6}{10} = \frac{3}{5}$ and $\frac{4}{10} = \frac{2}{5}$, $\frac{3}{5}$ and $\frac{2}{5}$ as well as any other fractional names for $\frac{6}{10}$ and $\frac{4}{10}$ respectively are also correct answers.

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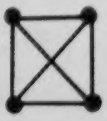
P04 **

Here are one red, one black and two blue marbles.



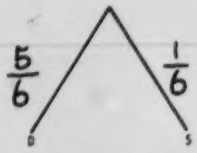
In the picture above show all possible ways to take two marbles.

How many ways can you take two marbles of the same color? 1

How many ways can you take two marbles of different colors? 5

What is the total number of ways of taking two marbles? 6

Write the probabilities next to the two lines in this picture:



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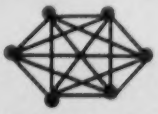
P05 **

Here are one red and five blue marbles.

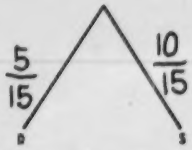


You mix them up and take two marbles without looking. Would you bet they are the same color or different colors? They are more likely to be of the same color.

What is the probability of getting different colors? $\frac{5}{15} = \frac{1}{3}$

What is the probability of getting the same color? $\frac{10}{15} = \frac{2}{3}$

Write the probabilities next to the two lines in this picture:*



* Since $\frac{1}{3} = \frac{5}{15}$ and $\frac{10}{15} = \frac{2}{3}$, $\frac{2}{6}$ and $\frac{4}{6}$ as well as any other fractional names for $\frac{1}{3}$ and $\frac{10}{15}$ respectively are also correct answers.

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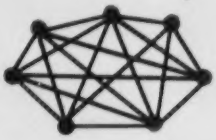
P06 ***

You have five blue and two red marbles. You mix them up and take two marbles without looking. Would you bet they are the same color or different colors? They are more likely to be of the same color.

What is the probability of getting different colors? $\frac{10}{21}$

What is the probability of getting the same color? $\frac{11}{21}$

(If you wish you may use the blank space below to draw any pictures or to make any calculations that may help you to answer the questions above.)



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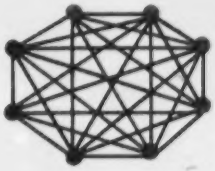
P07 ***

You have five blue marbles and three red marbles. You mix them up and take two marbles without looking. Would you bet they are the same color or different colors? They are more likely to be of the same color

What is the probability of getting different colors? $\frac{15}{28}$

What is the probability of getting the same color? $\frac{13}{28}$

(If you wish you may use the blank space below to draw any pictures or to make any calculations that may help you to answer the questions above.)



1.4 Same or Different #4

Begin this lesson with the following brief review of how probabilities are obtained. Show the children one red and two blue marbles.

T: If I choose one of these marbles without looking, what could I get?

S: You could get the red one or one of the two blue ones.

T: How many ways could I get a red marble?

S: One. You've only got one red marble.

T: How many marbles are there altogether that I could choose?

S: Three.

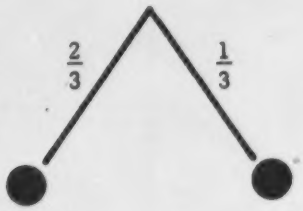
T: So what are the chances that I choose a red marble?

S: One chance out of three.

T: What is the probability of getting a red marble?

S: One-third.

Proceed similarly for the probability of getting a blue marble. Finish this brief review by drawing the following record of the situation:



T: So far, when we have played the "Same or Different" game we've used one red marble and at least two blue marbles. Do you think it's possible to play it with one red marble and only one blue marble?

Whatever the response, go ahead and play the game three or four times. Almost immediately your children will realize that you <u>always</u> get different colors.

T: As you've noticed, this game always turns out the same way. Here's what it looks like:

Draw the following on the board:

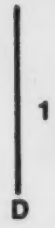
T: What is the probability for "D" in this game?

S: You've got one chance out of one, so the probability is $\frac{1}{1}$.

T: What's an easier way of writing "$\frac{1}{1}$"? What's 1 ÷ 1?

S: 1.

Record this on the board.



T: In this game we are <u>certain</u> that we will get different colors—it will happen every time you play the game. The probability of "D" is 1. Whenever anything is <u>certain</u> to happen its probability is 1. Do you think this is a very interesting game? Can you think of a more interesting game we could play with one red marble and one blue marble?

One of the suggestions you are likely to get will probably be as follows:

> Choose the marbles one at a time. That is, shake the two marbles, choose one at random and notice its color. Then shake the two marbles again, choose one at random again and notice whether its color is the same as or different from that of your first choice.

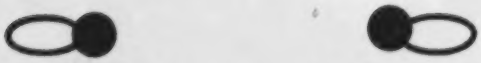
If no one suggests this, do so yourself. Play this game three or four times so that everyone understands how it is done. Point out that it is possible for the same marble to be chosen both times.

T: In a moment we're going to play this game, but first let's figure out what to expect. Let's find the probability of "D" and of "S". Here are the two marbles:

Draw the following on the board:

T: Show me how we could get the same color.

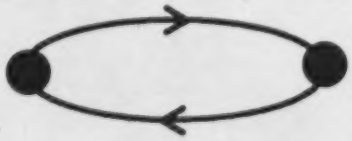
S: That will happen if you choose the same marble each time. You can show that by drawing loops.



The loop on the left means you choose the blue marble both times, and the other loop means you choose the red marble both times.

Beside this drawing, draw another copy of the two marbles.

T: Show me how we could get different colors.

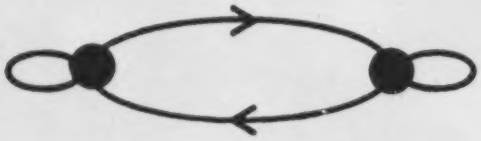


S: The top arrow means that you choose the blue marble first and the red marble second, and the bottom arrow means that you choose the red marble first and the blue marble second.

Be prepared for a student to suggest using the same method of recording that was used in the previous games.

If this happens, ask the student what it means. He or she will probably say that the colors are different. Then ask which marbles were chosen and in which order. The child will give one of the two possibilities mentioned above, say, blue first, red second. Then ask him or her how the other possibility could be shown. It will be realized that, in order to indicate in which order the marbles are chosen, an arrowhead is needed.

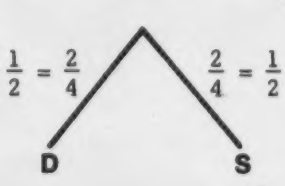
Finally, on a third drawing of the two marbles a child should show all the possibilities in this new game.



From this information the students will be able to tell you the probabilities of "D" and of "S", and then you should summarize by drawing the following diagram:

Bring out the fact that $\frac{2}{4} = \frac{1}{2}$ by referring to some analogy such as the dividing of a cake:

and extend your "probability tree" on the board:



T: Do you think this is a fair game?

S: Yes; both "D" and "S" have the same probability.

T: If we played the game 100 times about how often do you think we would get different colors? Why?

S: About 50 times, because "D" will occur about one-half of the time, and $\frac{1}{2} \times 100 = 50$.

Now divide the class into pairs, as in previous lessons, and tell the children that each pair is going to play the game ten times.

T: So how many games will we play altogether?

S: 150.*

T: About what do you think the results will be?

S: About 75 "S"'s and 75 "D"'s.

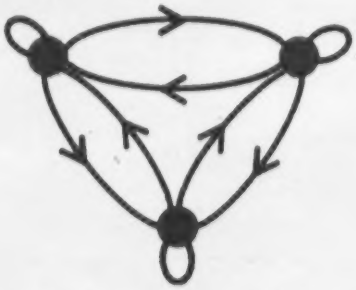
Let the children play. Record the data on the board as in previous lessons. Here are the results from a typical class of 30 students:

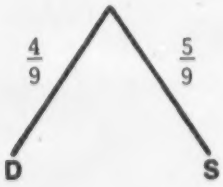
D	S
2	8
7	3
6	4
6	4
6	4
5	5
5	5
5	5
2	8
9	1
1	9
3	7
4	6
4	6
8	2
73	77

* In a "typical" class of 30 students.

Discuss with the class how closely the observed results agree with the calculated probable results.

Continue by considering the game in which you choose two marbles one at a time from among one red and <u>two</u> blue marbles, putting back the first choice marble before making the second choice. Proceed in the usual way until you have found the probabilities of "D" and "S".

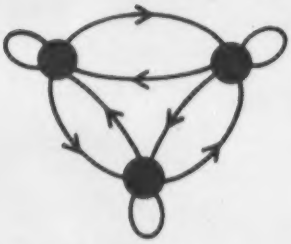


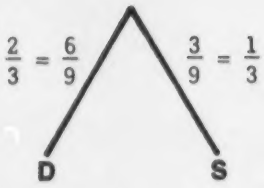


Ask the children to predict about what will happen if the game is played 90, 180, 200 times (a reasonable answer for this last one is: "about D: 90, S: 110" since the theoretical answer, D: $88\frac{8}{9}$, S: $111\frac{1}{9}$, has no practical significance).

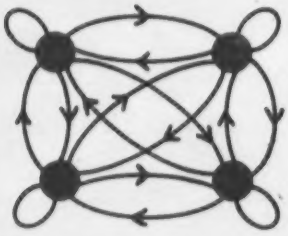
If time allows, you could go on to analyze one or both of the following two games. Either you can do the whole analysis on the board with the children's help, or you can do part on the board and let the children complete it individually, or you can let the children do the whole analysis by themselves. In each case, the game consists in choosing one marble at random from the given set of marbles, noticing its color, putting it back with the other marbles, choosing a second marble, and noticing whether it has the same color as the first choice or not.

1) One blue, one red, one green:





2) Two blue, two red:

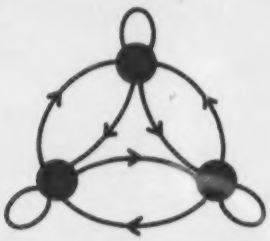


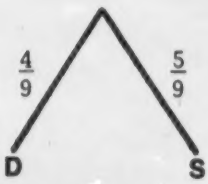
Notice that this second game is equivalent to the one with which we started the lesson in the sense that the "probability trees" of the two games are the same.

1.5 Same or Different #5

Begin the lesson by briefly reviewing the initial game of Section 1.4, choosing two marbles one at a time from among one red marble and one blue marble, replacing the first choice before making the second choice.

Then present the following game as if it were a new situation. This time you have one blue and two red marbles, and you choose two, one at a time, replacing the first before choosing the second. The analysis is as follows:



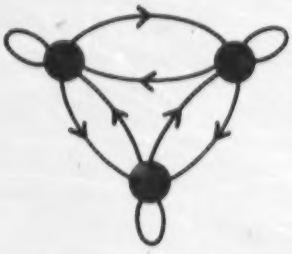


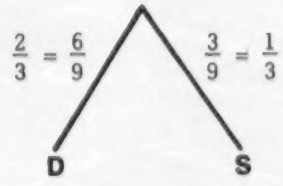
Someone will probably recognize that the probabilities of "D" and of "S" are exactly the same as in the game where you have one red and two blue marbles. (If no one does notice this, then go back over the analysis of that game and point out the similarity.) The important point here is to realize <u>why</u> the probabilities are the same. You may need to give a little assistance to a child who tries to explain this. Here is one possible way of seeing why the probabilities are the same:

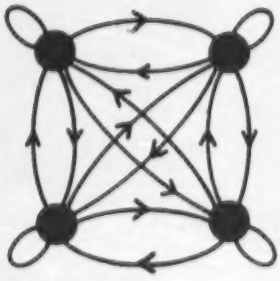
> In this game the two outcomes are "D" and "S". You're only interested in whether the colors are different or the same, <u>not</u> in what those colors actually are. So the important fact is that you have two marbles of one color and one of another; whether the two of the same color are red or blue makes no difference.

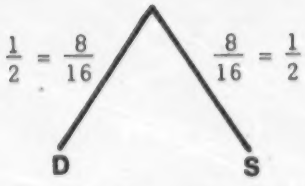
Continue by considering the two games mentioned at the end of Section 1.4 and completing any portion of the work on those games for which you did not

have sufficient time on the occasion of that lesson. If the analysis has yet to be done, let the children do this, working individually. You should start them merely by drawing the marbles on the board. Do not indicate any of the possible choices. Any child who completes this work successfully and rapidly should be asked to invent his or her own games of this type and analyze them. We reproduce below the analyses of these two games.









If you have not already done so, compare the second game to the game with one red marble and one blue marble.

Continue by asking the children what they think the probabilities for "D" and for "S" would be in the game with three blue and three red marbles. They will probably predict that they are D: $\frac{1}{2}$ and S: $\frac{1}{2}$ again (which is correct), but unless you're feeling particularly daring it would probably not be worth the trouble involved in drawing the picture showing all possible choices in order actually to show that this is the result.

T: Now let's go back again to the game with one red marble and one blue marble.

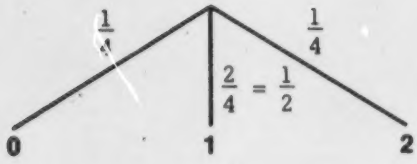
Up to now we've been interested in whether the two marbles we choose are the same color or different colors. Now we're going to change. I'm going to be interested in how many times I get a <u>blue</u> marble. How many times could I get a blue marble in this game?

S: Twice, if you choose the blue both times; or once, if you choose the blue one time and the red the other; or none at all, if you choose the red both times.

T: So now we're talking about a game with three possibilities.

Draw the following representation of the three possibilities on the board:

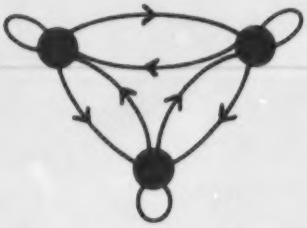
Discuss the situation with a view to finding the probabilities of each of these possibilities. For this purpose you will need to refer to the diagram which shows all the possible choices in this game (i.e., the one reproduced at the top of this page). From that diagram, it is clear that the probabilities are as shown in the following "probability tree":

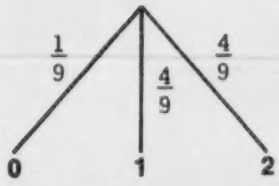


T: About what do you think would happen if we played this game 100 times?

S: You would choose two blue marbles about 25 times (because $\frac{1}{4} \times 100 = 25$); one blue marble about 50 times (because $\frac{1}{2} \times 100 = 50$ and no blue marbles at all about 25 times (because $\frac{1}{4} \times 100 = 25$).

If time and interest permit, conduct a similar discussion of the same modification of the game with two blue marbles and one red marble. Here are the results:





Finish the lesson by playing a modified version of the Shaker/Forecaster game you played during the lesson of Section 1.2. The modifications are as follows:

1) The Shaker is given one red marble, but may then request anywhere from one to five blue marbles.

2) The Shaker decides whether or not he or she will choose two marbles at once, or one at a time, replacing the first choice before making the second.

5P

Make sure that on each turn, the Forecaster is well informed about (a) how many blue marbles the Shaker is using, and (b) whether the Shaker is choosing with or without replacement.

2

THE SQUARE TRAP

Two games are introduced and explored in the storybook THE SQUARE TRAP. The random device used in the games is a spinner.* The emphasis is on the notion of expectation which arises through the process of averaging the outcomes of a large number of games. The approach is experimental. No systematic analysis of the games is attempted.

CAPSULE LESSON SUMMARIES

Section 2.1:	Introduce the storybook THE SQUARE TRAP and play the first of the games. Compile and examine the outcomes. (Worksheet P08)
Section 2.2:	Continue the examination in Section 2.1. (Worksheet P08)
Section 2.3:	Play and analyze the second game. You should find that the first and second game always end at the same time when they are played simultaneously. (Worksheet P08)

* NOTE ON SPINNERS: If your school happens to have available some commercially produced spinners whose faces are divided into two equal parts, feel free to use them in place of Worksheet P08 and the cardboard individual spinner arrows provided by CSMP.

2.1 "The Square Trap" #1

Give each student a copy of Worksheet P08, an individual spinner arrow, and a copy of the storybook THE SQUARE TRAP. Tell the children that they may read the story and explore the situation described therein as far as they like. When a child reaches pages 7 and 8, he or she should be encouraged to play the game, keeping a tally of what happens in the chart on page 8. If you have no checkers immediately available, the children can use a coin, an eraser, or some other relatively small object to keep track of which corner of the square they're on. If some of your children are particularly slow readers, you could organize them into a small group and read the story to them.

This individual work should last for at least one-half of the lesson. When you feel that the time is right, call the students' attention to yourself and ask for volunteers to tell that portion of the story that they have read. After discussing what is reported at this time, display one Minicomputer board with its face to the chalkboard. Place a magnetic checker at the lower left hand corner (this corner will correspond to "S" in the description of the game on page 5 of the storybook). Ask a student to explain how the game is played, and then to play a couple of games in front of the whole class, another child doing the necessary spinning with his or her spinner. Ask one more pair of students to play a couple of games in front of the class.

Divide the class into pairs and ask each pair to play the game at least ten times. Explain that they are to keep track of what happens by using the chart on page 8 of the storybook of one member of each pair as a tally-sheet. (Place a stroke in the appropriate column after each game.) If the chart on page 8 already contains the record of some games the child played in the first half of the lesson, the pair should just continue to add to it. While this is going on, copy the chart from page 8 onto the board. When most pairs have completed at least ten games, ask them to stop spinning and take part in a discussion of the results.

T: For how many of you was your checker trapped after exactly one move?

(Obviously, this should never have happened.)

T: For how many of you was your checker trapped after exactly two moves

(Find out from each pair how many times this occurred, and record the grand total for the class in the appropriate column.)

Continue in a similar way for the remaining columns of the chart. Here is the type of results that you can expect to have:

CHECKER TRAPPED AFTER										
1 move	2 moves	3 moves	4 moves	5 moves	6 moves	7 moves	8 moves	9 moves	10 moves	more than 10 moves
0	63	0	37	0	17	0	10	0	3	5

Your students will almost certainly have noticed that the checker can only be trapped after an even number of moves, never after an odd number of moves. Ask them to explain why this is so. Accept any correct explanation, no matter how awkwardly formulated. The children will also notice that the game ended many more times after two or four moves than it did after any larger number of moves.

Before you end the lesson, ask the children to write their names on their storybooks (because they will need their own books for the lessons of Sections 2.2 and 2.3). Then collect the books, the spinners, and the copies of Worksheet P08, and keep them ready for the next lesson on this topic.

2.2 "The Square Trap" #2

Before you give the children their copies of the storybook, go through the following preliminary exercises with the class.

Display one Minicomputer board with its face to the chalkboard and ask the children to recount what happened during the first lesson on THE SQUARE TRAP. Then continue by making the following statement:

T: At one point in the story, 0 took the spinner and spun the following sequence of digits: **111100001**. "My checker is trapped after nine moves," he shouted.

Let the students react to this statement, and explain that 0 was trying to trick the boy. (If your children have read the storybook, NABU WINS AN AWARD, they will know that the boy's name is "Nabu".)

T: Then 1 took a turn at playing the game. He spun **000110** and said, "My checker is trapped after six moves."

Let the children react to this statement, and then actually follow along that sequence of digits on the back of the Minicomputer board in order to show that in fact the checker was already trapped after four moves.

Proceed in a similar fashion for each of the situations described on page 10 of the storybook.

At this point, give out the copies of the books again and tell the children to open to pages 11 and 12 and read what happens next. Make sure that the children know what an <u>average score</u> is (it's the score you would have to get each time if you wanted to have the same total number of spins after getting the same score each game), and how to calculate it (in this case, the total

number of spins is 44, the number of games is ten; so, if you wanted to have the same score each game, you would have to answer the question "ten times what number is 44?" or equivalently, "what number is 44 divided by 10?").

Divide the class into pairs, distribute individual spinner arrows and copies of Worksheet P08, and tell each pair to play the game ten times (using the square on page 7, if necessary), keeping track of how many spins each game took. When they have played the ten games, they are to calculate their average score. As these results become available, record them on the board. In a class of 24, you might obtain results something like these:

3.2; 2.8; 4.8; 3.8; 3 ; 4.4;
5.2; 3.8; 4.3; 3.6; 3.8; 6.2.

T: What do you think about these average scores?

Someone will probably notice that 4.3 is not possible. Ten games with an average of 4.3 spins per game would give a total of 43 spins. But we already know each game must last an even number of moves, so the total number of spins in ten games must also be even. If you have any impossible average scores like this on your list, after discussing them, cross them off the list.

T: I wonder what the average score for the whole class is. How could we find out?

Someone will probably suggest adding up the average scores. Do that.

$3.2 + 2.8 + 4.8 + 3.8 + 3 + 4.4 + 5.2 + 3.8 + 3.6 + 3.8 + 6.2 = 44.6$

T: So altogether for 11 teams we got 44.6. What is the average score?

S: About 4, because $11 \times 4 = 44$.

T: Very good! Let's try to be a little more precise. Is the average score larger or smaller than 4?

S: Larger. We could try 4.1.

Write on the board:

$$11 \times 4.1 =$$

Ask students to calculate "11 × 4.1" and tell you the answer, explaining how they did the calculation.

$$11 \times 4.1 = 45.1$$

S: It's smaller than 4.1. It's almost exactly halfway between 4 and 4.1.

T: What number is halfway between 4 and 4.1?

S: 4.05.

Write on the board:

$$11 \times 4.05 =$$

Let the children calculate this number.

$$11 \times 4.05 = 44.55$$

Conclude that the class average is almost exactly 4.05.

T: Someone in another class got an average score of 2. What do you think about that?

S: It's possible, but it's not very likely. All ten games would have to have ended after only two spins.

T: Someone else in the same class got an average score of 1.8. What do you think about that?

S: That's impossible! They must have made a mistake. The smallest average score possible is 2.

Ask the children to turn the page and read pages 13 and 14. Discuss the long calculation our three friends made on page 14. Their checker was trapped 50 times in 180 spins, and they found that the average score was about 3.5.

T: Let's try to be a little more precise. Is the average score smaller or larger than 3.5?

Someone will probably suggest that you calculate "50 x 3.5". Ask the children to explain how this can be done and record the answer on the board.

$$50 \times 3.5 = 175.$$

S: So the average score is larger than 3.5. Let's try 3.6.

Write on the board:

$$50 \times 3.6 =$$

and let the students solve the problem by themselves.

$$50 \times 3.6 = 180.$$

So the average score is exactly 3.6.

Before you end the lesson remember to collect the storybooks, the individual spinner arrows, and the copies of Worksheet P08.

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Name _____ P08

SPINNER SHEET

SPINNER FACE
FOR
"THE SQUARE TRAP"

2.3 "The Square Trap" #3

Start the lesson by asking the children to give a brief account of what happened in the previous two lessons on this topic. Then distribute the storybooks, the individual spinner arrows, and copies of Worksheet P08. Ask the children to read pages 15 and 16 and to try to understand the game that I invented. While they are doing this, display two Minicomputer boards side by side, faces to the chalkboard.

When everyone has finished this reading, discuss the game together as a class. Place one blue and one red checker on the left hand Minicomputer board as in the picture on page 15, and ask a student to explain the game. With the help of another student (who is to do the necessary spinning), let him or her play the game once. Record (on the board) how many moves the game lasted. Let a couple of other pairs of students play the game once each while you record the results on the board. Ask the class to calculate the average score for this series of games.

Divide the class into pairs, and ask each pair to play the game ten times, recording their results by making tallies in the chart on page 17. When a pair of children have played their ten games, they should calculate their average score, and tell you so that you can record it on the board.

When everyone has finished playing their ten games, discuss what happened. Someone will certainly notice that this game also can only end after an even number of moves. Ask if anyone can explain this, and accept any correct explanation, no matter how awkwardly formulated. (One way of explaining this is as follows: you can pair all the moves, and you will thus obtain pairs of two kinds—(i) move a checker from left to right and then back again, (ii) move both checkers one after the other from left to right. The game is over as soon as a pair of moves of type (ii) appears. This will be preceded

by a certain number of pairs of moves of type (i). In any case, we're only talking about pairs of moves. so the total number of moves must be even.)

On the basis of the results recorded on the board, calculate the average score for the whole class, as in the lesson of Section 2.2. Then let the children read and answer the questions on page 18.

Display another Minicomputer board with its face to the chalkboard, separated somewhat from the other two you already have displayed. Place a checker at the lower left hand corner (which will correspond to "S", as shown in the picture on page 19).

T: Let's play both games at the same time.

Choose three volunteers: one to move the checker on the square; one to move the checkers between the two rooms; one to make the spins.

T: What game do you think will end first, 0's game or 1's game?

Accept without comment whatever the children have to say on this subject. Then let the volunteers play both games simultaneously once. They will both end at the same time. Repeat the simultaneous playing of the two games several times, using a different set of volunteers each time. It won't be long before the children realize that the two games are always ending at the same time, and some may even be able to explain why this must always be the case.

End the lesson by letting the students read the last page of the storybook. Encourage them to think of various games that could be played using the new spinner.